**CS 181 EXAM #2** — **NAME** ______________________________
**FALL 2015** — **UCLA ID** ______________________________

**You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.**

**1** Give a regular expression for each of the following languages:

(2 pts) **a.** strings of length at most 2015 over a given alphabet $\Sigma$;

(3 pts) **b.** binary strings in which the number of 0s and the number of 1s are not both odd.

**Solution.**

**a.** $(\Sigma \cup \varepsilon)^{2015}$

**b.** $1^*(01^*01^*)^* \cup 0^*(10^*10^*)^*$. There is a hard way to solve this problem and an easy one. The hard way is to construct an NFA and convert it to a regular expression. The easy way is to observe that this language is the union of two simpler languages: (i) binary strings with an even number of 0s, and (ii) binary strings with an even number of 1s.

(2 pts) **2** Simplify $(0 \cup 1)^*01(0 \cup 1)^* \cup 1^*0^*$ as much as possible, and explain your answer in detail.

**Solution.** The regular expression $(0 \cup 1)^*01(0 \cup 1)^*$ corresponds to binary strings that contain 01. The only strings that do not have this property are of the form $1^*0^*$. Thus, the union of $(0 \cup 1)^*01(0 \cup 1)^*$ and $1^*0^*$ simplifies to $(0 \cup 1)^*$.

(3 pts) **3** Prove or disprove: the Kleene star of every language is a regular language.

**Solution.** The claim is false. Consider $L = \{0^n 1^n : n \geqslant 0\}$. Then $L^*$ is the set of strings of the form $0^{n_1}1^{n_1}0^{n_2}1^{n_2}\cdots 0^{n_k}1^{n_k}$ for some $n_1, n_2, \ldots, n_k$. Let $p$ be arbitrary and consider the string $w = 0^p 1^p \in L^*$. If $x, y, z$ are strings such that $y$ is nonempty, $|xy| \leqslant p$, and $w = xyz$, then $xy^2z \notin L^*$. Therefore by the pumping lemma, $L^*$ is nonregular.

(3 pts) **4** Let $L$ be the language of palindromes over $\{0,1\}$. Determine the equivalence classes of $\equiv_L$ .

**Solution.** Let $u, v$ be arbitrary strings with $u \neq v$. Then for $N$ sufficiently large,

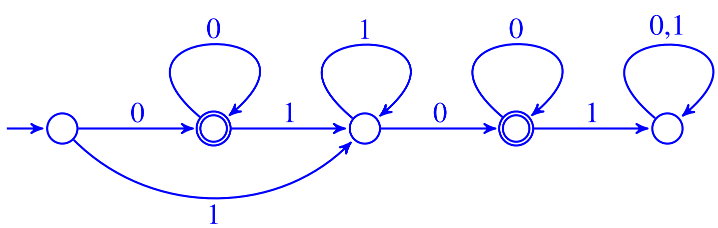
$$u\ \mathbf{01}^{N}\mathbf{0}\boldsymbol{u}^{\boldsymbol{R}} \in L,$$
$$v\ \mathbf{01}^{N}\mathbf{0}\boldsymbol{u}^{\boldsymbol{R}} \notin L.$$

Thus, every string is in an equivalence class by itself.

*Note.* A common mistake is to claim that $u\boldsymbol{u}^{\boldsymbol{R}} \in L, v\boldsymbol{u}^{\boldsymbol{R}} \notin L$ for any pair of distinct strings $u, v$. This claim fails for many string pairs, e.g., $0, 00$ as well as $0, 01010101$.

(4 pts) **5** Construct a DFA for $0^*1^*0^+$ with the smallest possible number of states. Prove that your DFA is the smallest possible.

**Solution.** The following DFA with five states recognizes $0^*1^*0^+$:




The five strings $\varepsilon, 0, 1, 10, 101$ are in pairwise distinct equivalence classes of $\equiv_{0^*1^*0^+}$, with distinguishing suffixes given by the following table:

| | $\varepsilon$ | 0 | 1 | 10 | 101 |
|---|---|---|---|---|---|
| $\varepsilon$ | | | | | |
| 0 | $\boldsymbol{\varepsilon}$ | | | | |
| 1 | **010** | **010** | | | |
| 10 | **10** | **10** | **10** | | |
| 101 | **0** | **0** | **0** | **0** | |

By the Myhill-Nerode theorem, we conclude that no smaller DFA exists.

**6** For each of the following languages, determine whether it is regular, and prove your answer:

(2 pts) **a.** binary strings in which the number of 1s is a multiple of the number of 0s;

(2 pts) **b.** binary strings of the form $uvuw$, where $u, v, w$ are strings and $u$ is nonempty;

(2 pts) **c.** nonempty binary strings of even length with the two middle symbols unequal;

(2 pts) **d.** strings over the alphabet $\{a, b, c\}$ that contain each of the three alphabet symbols.

**Solution.**

In each part, $L$ stands for the language in question.

**a.** Nonregular. For any positive integers $i < j$, we have $0^i\mathbf{1}^{\boldsymbol{i}} \in L$ but $0^j\mathbf{1}^{\boldsymbol{i}} \notin L$. Therefore, the strings $0, 00, 000, \ldots, 0^n, \ldots$ are each in a distinct equivalence class of $\equiv_L$. By the Myhill-Nerode theorem, $L$ is nonregular.

**b.** Regular. This language contains precisely those strings in which the first symbol occurs again, which corresponds to the regular expression $1\Sigma^*1\Sigma^* \cup 0\Sigma^*0\Sigma^*$.

**c.** Nonregular. For any even positive integers $i \neq j$, we have $0^i\mathbf{1}^{\boldsymbol{i}} \in L$ but $0^j\mathbf{1}^{\boldsymbol{i}} \notin L$. Therefore, the strings $0^2, 0^4, \ldots, 0^{2n}, \ldots$ are each in a distinct equivalence class of $\equiv_L$. By the Myhill-Nerode theorem, $L$ is nonregular.

**d.** Regular. The language is given by $\overline{\{a,b\}^* \cup \{a,c\}^* \cup \{b,c\}^*}$. Thus, it is obtained by applying the operations of Kleene star, union, and complement to the regular languages $\{a, b\}, \{a, c\}, \{b, c\}$. Since regular languages are closed under these operations, the result is regular.