NAME and UI[redacted] !!

CS180 — Algorithms and Complexity

Winter 2015

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Midterm Examination

OPEN BOOK, OPEN NOTES

Wednesday, February 11, 4:00–5:50pm

Do not cheat.

Problem	*Points*
1	22/25
2	25/25
3	25/25
4	15/25
Total	/100

In any answer on this exam, you can make reference to any definition or result from the [KT] text by giving a section number, page number, gray box number, verbal summary, etc.

There is NO need to provide a complete reproduction or proof of these results. Short answers are good.

Similarly, you can use any definition or result from the course notes, slides, homework assignments, etc. Just be clear in your references to these sources.

A **Master Theorem**: for $a > 1$, $b > 1$, $k \geq 0$, the solution for $T(n) = a\,T(n/b) + c\,n^k$ is

$$\begin{aligned} T(n) &= \Theta(\,n^{\ell}\,) && \text{if } k < \ell = \log_b a \\ T(n) &= \Theta(\,n^{\ell}\,\log n\,) && \text{if } k = \ell = \log_b a \\ T(n) &= \Theta(\,n^{k}\,) && \text{if } k > \ell = \log_b a. \end{aligned}$$

Minimum Spanning Tree in an undirected graph with edge costs $G = (V, E, c)$ a spanning tree T for G that has minimal total cost $\sum_{e \in T} c(e)$.

Shortest Path from a source node s to t in a directed or undirected graph $G = (V, E, \ell)$ h edge lengths ℓ is a path P from s to t with minimal total length $\sum_{e \in P} \ell(e)$. his exam, a **Shortest Path Tree** is a directed tree of edges outward from s selected by Dijkstra's algorithm.

ul identities: $$\sum_{k=1}^{N} k^p = \frac{1}{p+1} N^{p+1} + O(N^p) \qquad \sum_{k=1}^{N-1} a^k = \frac{a^N - 1}{a - 1} \qquad \sum_{k=1}^{N-1} k\,a^k = \frac{N\,a^N}{a-1} + O(a^N)$$

1. **The Master Theorem (25 points)** -3

Three platypuses meet in a bar and start to argue about the Master Theorem.

(a) **Master Theorem? (8 points)**

One of the platypuses says that, if we assume that $a > 1$, $b > 2$, $\ell = \log_b a$, and c and k are positive constants, then the recurrence $T(n) = a\,T(n/b) + cn^k$ has solution

$$T(n) = \begin{cases} \Theta(n^\ell) & \text{if } a > b^k \\ \Theta(n^k \log n) & \text{if } a = b^k \\ \Theta(n^k) & \text{if } a < b^k. \end{cases}$$

The other two platypuses laugh and say this is wrong. The first one gets angry and asks you to help prove it. The two laughing platypuses ask you to give a counterexample. What is your answer?

- [x] the first platypus is right, the recurrence is correct.
- [ ] the laughing platypuses are right, the recurrence is incorrect.

Short proof, or counterexample:

$T(n) = aT(n/b) + cn^k$

$T(n) = a^m T(1) + \ldots$

$T(n) = \ldots \Rightarrow \Theta(n^k \log n)$

$T(n) = \Theta(n^k)$

(b) **Laughing Theorem (8 points)**

One of the laughing platypuses says that the solution of $T(n) = a\,T(n/b) + c\log_b n$ is

$$T(n) = \Theta(n \log n) \qquad \text{(assuming } n \text{ is a power of } b\text{, and } T(1) = O(1))$$

and asks you to prove this. The angry platypus says no, and asks for the correct solution. What is your answer?

- [ ] the laughing platypus is right, the solution is correct.
- [x] the angry platypus is right, the solution is incorrect.

Short proof, or corrected solution:

$f(n) = c\log_b n \Rightarrow O(\log n)$ and $\Omega(\log n) \Rightarrow \Theta(\log n)$

$\Theta(\log n)$ is below $\Theta(n^\ell)$ (taken from Master Thm Def (slide 14/15 week))

therefore $T(n) = \Theta(n^\ell)$

3 (c) **Time Complexity (9 points)**

The platypuses start fighting over the asymptotic complexity $T(n)$ of the following algorithm A:

```
def A(x,y):
  if length(x) == 1:  return f(x,y); Θ(n)
  x1 = first_half(x); x2 = second_half(x); Θ(1)
  y1 = first_half(y); y2 = second_half(y); Θ(1)
  z1 = A( x1, y1 ); T(n/2)
  z2 = A( x2, y2 ); T(n/2)
  z3 = A( x1+x2, y1+y2 ); T(n/2)     } = 3T(n/2)
  return  z1 + z2 + z3 + f(x,y);
```

recurrence:	$T(n) = 3T(n/2) + \Theta(n)$
	$T(1) = \Theta(\ 1\ )$
solution:	$T(n) = \Theta(\ n\ )$

Assume that the inputs x and y are vectors of size n, and the input lengths n are always a power of 2. The functions `first_half` and `second_half` each take an vector of size n as input, and yield an output vector of size $n/2$. The function `f` takes time $\Theta(n)$ to compute if its arguments have length n.

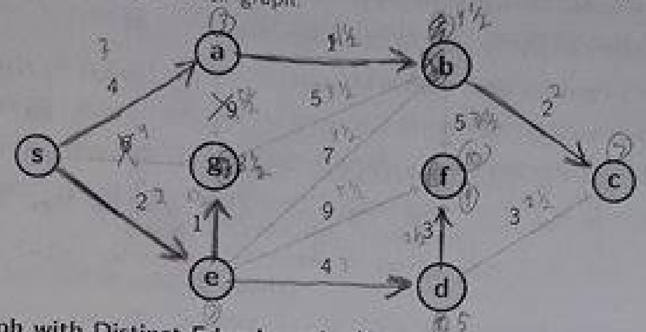
Please stop the fight by providing the recurrence for $T(n)$ and its solution in the box above.

$\log_3 2 < 1$

...rtest Paths (25 points) ✓

(a) Applying Dijkstra's algorithm (7 points)

Kim Kardashian and Kanye West buy the directed weighted graph G below for \$2M. They are trying to apply Dijkstra's algorithm to the graph, starting at vertex s. They want to know what is the order in which vertices get added to the shortest-path tree. Please tell them the order by filling in the box, and draw the resulting shortest-path tree on their graph.



order of addition to shortest-path tree:

1	2	3	4	5	6	7	8
s	e	g	a	b	d	c	f

(b) Graph with Distinct Edge Lengths (6 points)

Kim complains that their graph is too small and spends \$3M on a larger directed acyclic graph $G = (V, E, \ell)$. In this graph, all edges e have distinct (unique) lengths $\ell(e) > 0$. She asks you whether Dijkstra's algorithm is guaranteed to yield a unique shortest-path tree from any source node s in her new graph.

Your answer is: ■ Yes ☒ No. OK

Proof: If all edges are distinct then every time she runs the algorithm when the length of all edges stay the same she will get the same unique tree (shortest path) every time because Dijkstra's guarantees to always keep the path $P_{s,v}$ where v is the end of our unique shortest path the minimum at of all times.

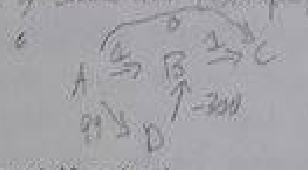
$\boxed{\text{Definition 4.14}}$

(c) Graph with Negative Edge Lengths (6 points)

Kanye has a life-changing experience and realizes *we all need negative edges*. He buys lots of directed graphs with negative edge lengths, but never buys graphs that have cycles with a negative total length. He asks you: "if I use Dijkstra's algorithm on these graphs, is its resulting shortest-path tree guaranteed to be correct?"

Your answer is: ☐ Yes ■ No. *Page 140 of [KT] also states this. 2nd paragraph down.

Proof: by contradiction (example)



our dijkstra's path will do

A → C = 0
A → B = 1
B → C = 1 (A→B→C = 2)
A → D → B = −201
finished. total = −199

This is an error because dist(c) should be −200 instead of 0 and total should be −200 not −199.

(d) Air Travel (6 points)

HW2 gives you the US airport graph $G = (V, E, \ell)$, and asks you to find a shortest path tree T from LAX. All edge lengths were given as distances in miles, but if we divide by some typical airspeed like 500mph, we can convert the edge lengths into hours. So assume each edge length $\ell(e)$ is given in hours.

Kim complains that your shortest paths are not realistic, since air travel requires at least a one-hour layover in each airport, so you change the length $\ell(e)$ of every edge e in the graph to $\ell'(e) = \ell(e) + 1$ hour.

Is the tree T for G guaranteed to still be a shortest-path tree from LAX for the changed graph $G' = (V, E, \ell')$?

Your answer is: ☐ Yes ■ No.

Proof: The only time changing the weight of all edges will not effect the shortest path is if it is changed linearly. Converting the length to hours is done linearly but adding the extra hour is not done linearly and can cause a change in the shortest path tree.

3. Minimal Spanning Trees (25 points)

Two highly-paid consultants, Kleinberg and Tardos, are arguing about spanning trees in an undirected graph G. You are hired as an even more highly-paid consultant-consultant to resolve their dispute.

(a) MSTs with Negative Costs (9 points)

Assume that all edge costs c are distinct, but the edge costs are permitted to be negative.

Kleinberg argues the MST can be determined just by using Kruskal's algorithm.

Tardos says this is ridiculous, Kruskal's algorithm won't work in this case.

Which consultant is right? ■ Kleinberg ☐ Tardos

Proof:

Kruskal's simply adds the edges in order of ascending weight. negative values are less than positive and more negative (-5) is still less than less negative (-1) edges. Therefore we will still add edges in the same ascending order as without negative edges.

(b) Maximum Spanning Trees (9 points)

The company changes its specifications so that all edge costs c must satisfy $c > 1$, and it wants an algorithm to construct **Maximum** Spanning Trees. In other words, it wants an efficient algorithm that finds a spanning tree with the property that the sum of its edge costs is *maximum*.

Kleinberg says that this new Maximum Spanning Tree problem is hard, and will take exponential time to solve.

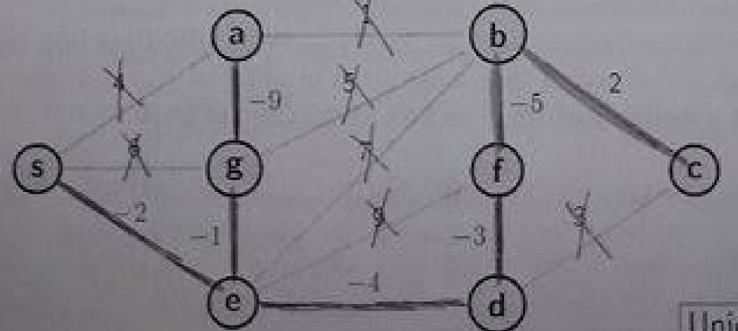
Tardos says the problem can be easily solved with minor changes to any Minimum Spanning Tree algorithm.

Which consultant is right? ☐ Kleinberg ■ Tardos

Proof: We can take the reverse delete algorithm, which removes the highest weighted edge first, then continue in descending order unless it disconnects G, and instead remove the lowest weighted edge first, then continue in ascending order unless it disconnects G. This is a very easy/simple change to the algorithm and works perfectly.

(c) Overpaid Consultants (7 points)

Kleinberg and Tardos cannot figure out a MST for the following graph. Please draw a MST for them, and tell them whether or not it is unique.



Unique MST? ■ Yes ☐ No

all unique weights

...rview Questions (25 points) −10

✓ (a) Changing Minimum Spanning Trees (3 points)

For an undirected graph G with distinct (unique) edge costs, which of the following statements are true?

True ☐	False ■	The MST could change if we change the cost of each edge e from $c(e)$ to $c(e)+2$.
True ☐	False ■	The MST could change if we change the cost of each edge e from $c(e)$ to $2c(e)$.
True ☐	False ■	The MST could change if we change the cost of each edge e from $c(e)$ to $c(e)^2$.

✓ (b) DAGs (6 points)

True ■	False ☐	A directed graph is a directed acyclic graph if and only if it can be topologically sorted.
True ☐	False ■	G is a directed acyclic graph if and only if G has a node with no incoming edges.
True ☐	False ■	A 'DFS tree' T starting from node s in an undirected graph G is sometimes a directed aylic graph that is not a tree.

−2 (c) Undirected Graphs (8 points)

True ☐	False ■	Suppose $G=(V,E)$ is an undirected, connected graph in which all vertices have even degree. Then G is bipartite.
True ■	False ☒	G is a bipartite graph if and only if G has no triangles (i.e., no three nodes are a clique).
True ☐	False ■	Suppose a weighted undirected graph G has a cycle C, and there is an edge e that is the unique least-cost edge in C. Then e is in every MST for G.
True ■	False ☐	If all edge lengths in an undirected graph G are a constant $c>0$, then for every source vertex s in G, a shortest path tree from s is the same as a BFS tree starting from s.

−4 (d) Longest Path Problem (4 points)

In the longest path problem, we're given a weighted directed graph $G=(V,E,\ell)$, and a source $s \in V$, and we're asked to find the longest path from s to every vertex in G.
In general, it's not known whether there is an efficient algorithm to solve the Longest Path problem.
If we restrict G to be acyclic, however, this problem can be solved in polynomial time.
Give an efficient algorithm for finding the Longest Paths from s in a weighted directed acyclic graph G.
(Hint: no incoming edges)

~~If we use our maximum spanning tree reverse-delete algorithm from 3b, we first do this to our graph. This gives us all our max edges in a directed acyclic graph. We can then traverse through the graph starting at s and add our longest edge to our set S. We can continue doing this until we hit every node that can reach our end node, producing the longest path.~~

greed won't work here
− it can be fooled.

−4 (e) Hamiltonian Path (4 points)

A Hamiltonian path is a path that visits all nodes in a graph. Explain how, given a *directed acyclic graph G*, it is possible to determine in time $O(V+E)$ whether G has a Hamiltonian path.

~~Yes it is. An example would be to do a ~~DFS~~ algorithm on G and use an adjacency list representation which is O(m+n) where m = V and n = E. Taken from 3.13 Definition. DFS tree will visit all possible nodes in O(V+E) time and then check if there are any nodes in G not visited yet. If there is a node not visited, there is no Hamiltonian cycle. If no node exists, then a Hamiltonian cycle exists.~~ ?

~~~~ BFS

?
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