NAME and UID: ______________________________ **!!**

CS170A – Mathematical Models and Methods for Computer Science
Spring 2013
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Sample Midterm Examination
OPEN BOOK, OPEN NOTES, CLOSED NETWORKING
Wednesday, May 8, 2013, 4:00am-5:50am

Do not cheat.

Problem	*Points*
1	/25
2	/25
3	/25
4	/25
Total	/100

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1. **SVD**

For each of the following matrices, determine (yes/no) whether the matrix is: unitary, hermitian, invertible, positive definite. (Recall A is positive definite if it is Hermitian and all of its eigenvalues are positive real values.) Assume that $i = \sqrt{-1}$.

matrix	unitary?	hermitian?	invertible?	positive definite?
$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐
$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐
$\begin{pmatrix} i & i \\ i & 0 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐
$\begin{pmatrix} i & -i \\ -i & i \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐

For each of the following equations, mark whether the equation is True (valid for all specified matrices X) or False. Assume that X' denotes the transpose of X, X' denotes the hermitian transpose of X, and $i = \sqrt{-1}$.

For full credit, if you mark it True, you must show how you derived this. If you mark it False, you must give a *counterexample* (values for these sets for which the equation does not hold).

(a) **True** ☐ **False** ☐
The SVD $X = U\,S\,V'$ of any 2×2 rotation or reflection matrix X has $U = X$ and $S = V' = I$, where I is the identity matrix.

(b) **True** ☐ **False** ☐
If X has the SVD $X = U\ S\ V'$, where S is the diagonal matrix of singular values of X, and U and V are unitary matrices, then the solution of the least squares problem $X\mathbf{c} = \mathbf{y}$ is $\mathbf{c} = V\ S^{-}\ U'\ \mathbf{y}$, where S^{-} is the pseudoinverse of S.

(c) **True** ☐ **False** ☐
If X has the SVD $X = U\ S\ U'$, where S is the diagonal matrix of singular values of X and U is a real unitary matrix, then X is real symmetric and U is orthogonal.

(d) **True** ☐ **False** ☐
If X has the SVD $X = U\ S\ U'$, where S is the diagonal matrix of singular values of X and U is a real unitary matrix, and X is invertible, then X is positive definite.

(e) **True** ☐ **False** ☐
If X is a square matrix and the SVD $X = U\,S\,V'$ of X has only positive values on the diagonal of S, then X is positive definite.

(f) **True** ☐ **False** ☐
If X is a covariance matrix and its SVD is $X = U\,S\,V'$, then $U = V$ and all diagonal entries in S are positive.

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2. **Least Squares**
Gauss became famous because he was able to fit an ellipse through the observed positions of the asteroid Ceres using least squares. Your job here is to show how obtained this fit.

Suppose you are given a table of n pairs of real (x, y) values

x	y
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

and you are asked to fit the line

$$\alpha\,x \;+\; \beta\,y \;=\; 1$$

through this data, finding real coefficients a and b that minimize squared error.

(a) Express this as a linear problem. That is, give a matrix X whose entries are functions of $\{x_i\}$ and $\{y_i\}$ for which α and β can be obtained by least squares on $X \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ vector whose entries are all 1.

(b) Give an explicit formula for $X'X$ (in terms of $\{x_i\}$ and $\{y_i\}$).

(c) Give an explicit formula for α (in terms of $\{x_i\}$ and $\{y_i\}$).

3. **Covariance**

Assume that C is the $n \times n$ covariance matrix $\mathrm{cov}(X)$, so

$$C \;=\; \mathrm{cov}(X) \;=\; 1/(n-1)\;\; (X-\bar{X})'\,(X-\bar{X})$$

where $\bar{X}$ is the matrix whose columns are filled with means of corresponding columns in X.

(a) Suppose $(X-\bar{X})$ has the SVD

$$(X-\bar{X}) \;=\; U\;S\;V'$$

where $S \;=\; diag(s_1,\ldots,s_n)$ is a diagonal matrix of singular values; U and V are unitary. Show then that C must have the SVD

$$C \;=\; V\;\;(1/(n-1)\;\;S^2)\;V'.$$

(b) If we increase every value in the matrix X by adding a constant a, exactly how will its covariance matrix $\mathrm{cov}(X)$ change?

(c) Assume that $S \;=\; diag(s_1,\ldots,s_n)$ is invertible, so all the singular values are non-zero. What is the SVD of C^{-1}?

(d) For the covariance matrix C with SVD

$$C \;=\; \begin{pmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{pmatrix} \;=\; \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

find the principal components of C.

4. **PCA**
Consider the following list of polyhedra and their properties:

Polyhedron	Faces	Vertices	Edges
tetrahedron	4	4	6
octahedron	8	6	12
cube	6	8	12
dodecahedron	12	20	30
icosahedron	20	12	30

Suppose we want to find a linear relationship between the number of Faces, Vertices, and Edges. That is, we want to find real values α, β, γ such that

$$Faces \quad = \quad \alpha \cdot Vertices \quad + \quad \beta \cdot Edges \quad + \quad \gamma$$

(a) Convert the table above into a set of 5 equations involving the variables α, β, and γ.

(b) Explain how these equations could be solved using least squares, by converting them to matrix form and explaining what matrix operations you could use to find α, β, and γ.

(c) Suppose that

$$X = \begin{pmatrix} 4 & 6 & 1 \\ 6 & 12 & 1 \\ 8 & 12 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix}.$$

Verify that $X\mathbf{c} = \mathbf{y}$ expresses the relationship between Faces, Edges, and Vertices. How can we solve this system for the values α, β, and γ?

(d) It turns out that

$$cov(X) = \begin{pmatrix} 4 & 6 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

How many principal components with nonzero eigenvalues does X have?