CS112: Computer System Modeling Fundamentals
Final Solutions 2011

Problem 1 (4 points)

A virus is present on 20% of the computers in the CS department. A particular test for the virus has a false positive rate (i.e., percentage of the time that the test will come back positive when the computer *doesn't* have the virus) of 10% and a false negative rate (i.e., percentage of the time that the test will come back negative when the computer *does* have the virus) of 40%. If a particular computer tests positive for the virus, what is the probability that this computer actually has the virus?

Solution

Let V be the event that the computer has the virus.
Let T be the event that the test comes back positive.

$$\textbf{Given:}\quad \begin{array}{ll} \mathbf{P}(V) = 20\% & \mathbf{P}(\neg V) = 80\% \\ \mathbf{P}(T|\neg V) = 10\% & \mathbf{P}(T|V) = 60\% \\ \mathbf{P}(\neg T|\neg V) = 90\% & \mathbf{P}(\neg T|V) = 40\% \end{array}$$

$$\begin{aligned} \mathbf{P}(V|T) &= \frac{\mathbf{P}(V)\mathbf{P}(T|V)}{\mathbf{P}(T)} \\ &= \frac{\mathbf{P}(V)\mathbf{P}(T|V)}{\mathbf{P}(T|V)\mathbf{P}(V) + \mathbf{P}(T|\neg V)\mathbf{P}(\neg V)} \\ &= \frac{20\% \times 60\%}{60\% \times 20\% + 10\% \times 80\%} \\ &= \frac{20\% \times 60\%}{60\% \times 20\% + 10\% \times 80\%} = \frac{12\%}{12\% + 8\%} = 60\% \end{aligned}$$

Problem 2 (15 points)

Let X be a random variable with PDF

$$f_X(x) = \begin{cases} x/4 & \text{if } 1 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and let A be the event $X \geq 2$.

a) Find $\mathbf{E}[X]$.

b) Find $\mathbf{P}(A)$.

c) Find $f_{X|A}(x)$.

d) Find $\mathbf{E}[X|A]$.

e) Find $\mathbf{Var}(X)$. (*Hint: The easiest solution uses your answer from part a.*)

Solution

a)

$$\mathbf{E}[X] = \int_1^3 x \times \frac{x}{4}\, dx \quad = \quad \frac{1}{4}\int_1^3 x^2\, dx \quad = \quad \frac{1}{4}\frac{x^3}{3}\Big|_{x=1}^{x=3} \quad = \quad \frac{27}{12} - \frac{1}{12} \quad = \quad \frac{26}{12} \quad = \quad \frac{13}{6}$$

b)

$$\mathbf{P}[A] = \int_2^3 \frac{x}{4}\, dx \quad = \quad \frac{1}{4}\int_2^3 x\, dx \quad = \quad \frac{1}{4}\frac{x^2}{2}\Big|_{x=2}^{x=3} \quad = \quad \frac{9}{8} - \frac{4}{8} \quad = \quad \frac{5}{8}$$

c)

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{\mathbf{P}(A)} & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{\frac{x}{4}}{\frac{5}{8}} & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2x}{5} & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

d)

$$\begin{aligned}\mathbf{E}[X|A] &= \int_2^3 x\, f_{X|A}(x)\, dx \quad = \quad \int_1^3 x \times \frac{2x}{5}\, dx \quad = \quad \frac{2}{5}\int_2^3 x^2\, dx \quad = \quad \frac{2}{5}\frac{x^3}{3}\Big|_{x=2}^{x=3} \\ &= \frac{54}{15} - \frac{16}{15} \quad = \quad \frac{38}{15}\end{aligned}$$

e)

$$\mathbf{E}[X^2] = \int_{x=1}^3 x^2 \frac{x}{4}\ dx = \frac{1}{4}\int_{x=1}^3 x^3\ dx = \frac{1}{4}\left[\frac{x^4}{4}\Big|_{x=1}^3\right] = \frac{1}{4}\left[\frac{81}{4} - \frac{1}{4}\right] = \frac{80}{16} = 5$$

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - (E[X])^2 = 5 - \frac{13}{6}^2 = \frac{180}{36} - \frac{169}{36} = \frac{11}{36}$$

Problem 3 (6 points)

You arrive at the bank to find either zero, one, or two customers in front of you with probability p_0, p_1, and p_2 respectively. The time it takes to service a customer is an exponential random variable with parameter λ. What is the expected amount of time you must wait before you get to the front of the line?

Solution

Let W be your wait time.
Let N be the number of customers in line before you.
Let T be exponential random variable with paramter λ representing the time it takes to service a customer.
$\mathbf{E}[T] = \frac{1}{\lambda}$

Using the law of total expectation:

$$\begin{aligned}
\mathbf{E}[W] &= \mathbf{E}[W|N=0] \times \mathbf{P}[N=0] + \mathbf{E}[W|N=1] \times \mathbf{P}[N=1] + \mathbf{E}[W|N=2] \times \mathbf{P}[N=2] \\
&= 0 \times p_0 + \frac{1}{\lambda} \times p_1 + \frac{2}{\lambda} \times p_2 \\
&= \frac{1}{\lambda}(p_1 + 2p_2)
\end{aligned}$$

Problem 4 (11 points)

Knowing that snow is extremely rare in Los Angeles, you believe that the probability of there being snow outside on a given day in January is 1/10,000. However, when you wake up one January morning, your roommate tells you that there is snow outside. Your roommate is very trustworthy: if there really is snow, he will definitely tell you there is snow. If there is not snow, he will tell you that there is snow with probability 0.01.

a) With what probability do you now think there is snow outside? (You may leave your answer in terms of arithmetic operations, such as addition and multiplication.)

b) What is the maximum likelihood hypothesis (snow or not snow)? (Justify your answer mathematically.)

c) What is the MAP hypothesis? (Again, justify your answer mathematically.)

Solution

Let S be the event there is snow outside.
Let T be the event that your roommate tells you there is snow outside.

Given:

$$\begin{array}{ll}
\mathbf{P}(S) = \frac{1}{10{,}000} & \mathbf{P}(\neg S) = \frac{9{,}999}{10{,}000} \\
\mathbf{P}(T|S) = 1 & \mathbf{P}(\neg T|S) = 0 \\
\mathbf{P}(T|\neg S) = \frac{1}{100} & \mathbf{P}(\neg T|\neg S) = \frac{99}{100}
\end{array}$$

a)

$$\mathbf{P}(S|T) = \frac{\mathbf{P}(S)\mathbf{P}(T|S)}{\mathbf{P}(T)} = \frac{\mathbf{P}(S)\mathbf{P}(T|S)}{\mathbf{P}(S)\mathbf{P}(T|S) + \mathbf{P}(\neg S)\mathbf{P}(T|\neg S)}$$
$$= \frac{\frac{1}{10{,}000} \times 1}{\left(\frac{1}{10{,}000} \times 1\right) + \left(\frac{9{,}999}{10{,}000} \times \frac{1}{100}\right)} = \frac{\frac{1}{10{,}000}}{\frac{1}{10{,}000} + \frac{9{,}999}{1{,}000{,}000}} = \frac{1}{1 + \frac{9{,}999}{100}} = \frac{100}{10,099}$$

b) Let H^{ML} be our maximum likelihood hypothesis.
Let S_1 be the hypothesis that it is snowing.
Let S_0 be the hypothesis that it is not snowing.

$$H^{ML} = \arg\max_i \mathbf{P}(T|S_i)$$
$$\mathbf{P}(T|S_1) = 1$$
$$\mathbf{P}(T|S_0) = \frac{1}{100}$$

Therefore, $H^{ML} = S_1$, i.e., that it is snowing.

c) Let H^{MAP} be our maximum a posteriori hypothesis.

$$H^{MAP} = \arg\max_i \mathbf{P}(T|S_i)\mathbf{P}(S_i)$$
$$\mathbf{P}(T|S_1)\mathbf{P}(S_1) = 1 \times \frac{1}{10,000} = \frac{100}{1,000,000}$$
$$\mathbf{P}(T|S_0)\mathbf{P}(S_0) = \frac{1}{100}\frac{9,999}{10,000} = \frac{9,999}{1,000,000}$$

Therefore, $H^{MAP} = S_0$, i.e., that it is not snowing.

Problem 5 (7 points)

A new food truck allows students to buy either a sandwich or a wrap, but not both. Sandwiches must contain either bacon or turkey, but not both. Wraps contain turkey. There are 3 extra ingredients {lettuce, tomato, mayo} that can be combined with bacon and 4 extra ingredients {cheese, mustard, cucumber, peppers} that can be combined with turkey. Any sandwich can have any subset (including the empty subset) of the available extras. A wrap can have at most 2 extra ingredients.

a) How many possible ways are there to order a bacon sandwich?

b) How many ways are there to order a turkey wrap?

c) How many possible lunch options are there? For example, possible lunch options include "a bacon sandwich with lettuce, tomato, and mayo," "a turkey wrap with cheese," and "a turkey sandwich with no extras."

Solution

a) There are 3 extra ingredients that can be combined with bacon sandwiches. Therefore, there are $2^3 = 8$ possible bacon sandwiches.

b) There are 4 ingredients that can be combined with turkey wraps and we can choose up to 2 of them. Therefore, there are $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 1 + \frac{4!}{3!1!} + \frac{4!}{2!2!} = 1 + 4 + 6 = 11$ possible turkey wraps.

c) We have already calculated all possible bacon sandwiches and turkey wraps, the only other possibility is a turkey sandwich. There are 4 ingredients that can be combined with turkey sandwiches. Therefore, there are $2^4 = 16$ possible turkey sandwiches. Adding this to the answers to parts a and b, we have 8 bacon sandwiches + 11 turkey wraps + 16 turkey sandwiches = 35 different lunch options.

Problem 6 (10 points)

Suppose 10% percent of my email is from my friend Hanna, who really likes to talk about food. There are 4 words that are particularly indicative of an email being from Hanna: "steak", "delicious", "bacon", and "cheese". For any given email I therefore have 4 pieces of data—the presence or absence of each of these words. I also know:

$$\begin{array}{ll}
\mathbf{P}(\text{steak} \mid \text{Hanna}) = 0.8 & \mathbf{P}(\text{steak} \mid \neg\,\text{Hanna}) = 0.1 \\
\mathbf{P}(\text{delicious} \mid \text{Hanna}) = 0.9 & \mathbf{P}(\text{delicious} \mid \neg\,\text{Hanna}) = 0.1 \\
\mathbf{P}(\text{bacon} \mid \text{Hanna}) = 0.3 & \mathbf{P}(\text{bacon} \mid \neg\,\text{Hanna}) = 0.05 \\
\mathbf{P}(\text{cheese} \mid \text{Hanna}) = 0.7 & \mathbf{P}(\text{cheese} \mid \neg\,\text{Hanna}) = 0.02
\end{array}$$

a) Draw the graphical model representing the conditional independence assumptions implied by Naive Bayes for this scenario. Annotate your model with the relevant conditional probability tables.

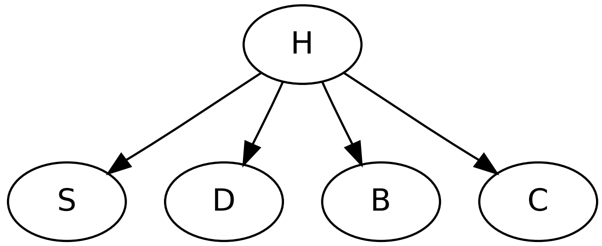
b) Use the Naive Bayes assumption to determine whether an email that contains the words "delicious", "bacon", and "cheese", but not the word "steak", should be classified as being from Hanna according to the MAP rule.

Solution

a) Let:

- S be whether or not the email contains the word steak.
- D be whether or not the email contains the word delicious.
- B be whether or not the email contains the word bacon.
- C be whether or not the email contains the word cheese.
- H be whether or not the email is from Hanna.

H	$\neg H$
.1	.9



	S	$\neg S$	D	$\neg D$	B	$\neg B$	C	$\neg C$
H	.8	.2	.9	.1	.3	.7	.7	.3
$\neg H$	.1	.9	.1	.9	.05	.95	.02	.98

b)

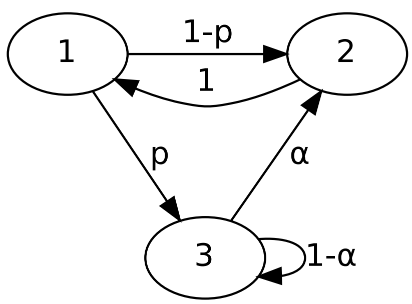
$$\mathbf{P}(\neg S, D, B, C|H) \approx \mathbf{P}(\neg S|H) \times \mathbf{P}(D|H) \times \mathbf{P}(B|H) \times \mathbf{P}(C|H) = .2 \times .9 \times .3 \times .7 = .0378$$
$$\mathbf{P}(\neg S, D, B, C|\neg H) \approx \mathbf{P}(\neg S|\neg H) \times \mathbf{P}(D|\neg H) \times \mathbf{P}(B|\neg H) \times \mathbf{P}(C|\neg H) = .9 \times .1 \times .05 \times .02 = .00009$$

$$\mathbf{P}(\neg S, D, B, C|H)\mathbf{P}(H) \approx .0378 \times 0.1 = 0.00378$$
$$\mathbf{P}(\neg S, D, B, C|\neg H)\mathbf{P}(\neg H) \approx .00009 \times 0.9 = 0.000081$$

And so the naive Bayes classification is H, that the email is from Hanna.

Problem 7 (8 points)

Consider the following Markov Chain state diagram:



a) Find P, the probability transition matrix.

b) Under what constraints on α and p will the Markov chain have exactly one recurrent class and satisfy the property that the class is not periodic? (It might be easier to list the conditions under which this does *not* hold.)

c) Solve for the steady state probabilities π_1, π_2, and π_3 in terms of p and α.

Solution

a)

$$\mathbf{P} = \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \end{bmatrix}$$

b) If $\alpha = p = 1$, then the Markov chain will be periodic with period 3.

If $p = \alpha = 0$, the Markov chain will have two recurrent classes.

If $p = 0$, then the Markov chain will be periodic with period 2.

The Markov Chain will have one recurrent class and will be aperiodic for all other values of α and p between 0 and 1.

c) We solve for $\pi = \pi\mathbf{P}$.

$$\begin{aligned} \pi_1 &= \pi_2 \\ \pi_2 &= (1-p)\pi_1 + \alpha\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

Thus, $\pi_1 = \pi_2 = \frac{\alpha}{p+2\alpha}$
and $\pi_3 = \frac{p}{p+2\alpha}$

Problem 8 (9 points)

A statistician would like to estimate the number of cars owned by each household in a population, based on n independent samples $X_1, \ldots, X_n$ chosen uniformly from the population (i.e., each X_i is the number of cars of a randomly chosen *household*, where each household is equally likely to be chosen). She uses the sample mean M_n as her estimate. She guesses that the standard deviation of the samples could not be more than 2. For the following questions, assume that this guess is correct.

a) What is the smallest that n could be while guaranteeing that the standard deviation of M_n is no more than 0.1?

b) What is the smallest that n could be if we would like Chebyshev's inequality to guarantee that M_n is within 0.1 of the true mean number of cars with probability at least 0.95?

Solution

a) Since the samples $X_1, \ldots, X_n$ are independent, we have that

$$\mathbf{Var}(M_n) = \mathbf{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right)^2 \mathbf{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathbf{Var}\left(X_i\right) \leq \frac{4n}{n^2} = \frac{4}{n}.$$

The inequality follows from the fact that the statistician is assuming that the standard deviation of the samples could not be more than 2, and therefore the variance of each X_i is no more than $2^2 = 4$.

This implies that the standard deviation of M_n is upper bounded by $2/\sqrt{n}$. Therefore, if $2/\sqrt{n} \leq 0.1$, then the standard deviation will be no more than 0.1. This is guaranteed to be true if $\sqrt{n} \geq 20$, or $n \geq 400$.

b) First note that $\mathrm{E}[M_n]$ is the true average number of cars. Call this average r. Then we are interested in determining the minimal value of n such that Chebyshev's Inequality implies that

$$\mathbf{P}(|M_n - r| \leq 0.1) \geq 0.95,$$

or equivalently,

$$\mathbf{P}(|M_n - r| \geq 0.1) \leq 0.05.$$

Chebyshev's Inequality tells us that

$$\mathbf{P}(|M_n - r| \geq 0.1) \leq \frac{\sigma_n^2}{0.01}$$

where σ_n^2 is the variance of M_n. We know from part (a) that

$$\sigma_n^2 = \mathbf{Var}(M_n) \leq \frac{4}{n}$$

which tells us that

$$\mathbf{P}(|M_n - r| \geq 0.1) \leq \frac{4}{0.01n} = \frac{400}{n}.$$

Setting $n = 8000$ gives us the result we want.

Problem 9 (16 points)

Your friend has 2 super-cute gray kittens. One kitten is harmless: when you feed him a treat, he never bites. The other kitten is mean: when you feed him a treat, he bites with probability 1/2 (regardless of whether he's bitten previously). Unfortunately, the kittens are identical twins. You don't know which is which.

Make sure that you clearly define any events or random variables used in your solutions.

a) Suppose you pick a kitten at random (i.e., the probability of choosing each kitten is 1/2) and then feed him *two* treats. What is the probability that you emerge unscathed (i.e., that you get no bites)?

b) Suppose you pick a kitten at random and feed him a treat. You observe that you do not get bitten. Given this observation, what is the probability that the kitten you chose is harmless?

c) You would like to determine whether the kitten chosen in part b is harmless or mean. What is the MAP hypothesis?

d) Suppose you pick a kitten at random. You feed him *two* treats and do not get bitten either time. Given this, what's the probability that when you feed him a third treat you will get bitten?

Solution

Let K_H be the event that you choose the harmless kitty. $\mathbf{P}(K_H) = \frac{1}{2}$
Let K_M be the event that you choose the mean kitty. $\mathbf{P}(K_M) = \frac{1}{2}$
Let B be the event that we are bit and $\neg B$ be the event we are unscathed. Let C be the event that we are bit and $\neg C$ be the event we are unscathed, when we feed twice.

a) If we choose the harmless kitty, $\mathbf{P}(\neg C|K_H) = 1$ regardless of how many treats we feed him. If we choose the mean kitty, $\mathbf{P}(\neg C|K_M) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ since there is a $\frac{1}{2}$ probability of not being bit with each treat fed. Therefore, using the law of total probability:

$$\mathbf{P}(\neg C) = \mathbf{P}(\neg C|K_H)\mathbf{P}(K_H) + \mathbf{P}(\neg C|K_M)\mathbf{P}(K_M) = \left(1 \times \frac{1}{2}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{5}{8}$$

b) Given that we feed our kitten 1 treat, we have $\mathbf{P}(\neg B|K_M) = \frac{1}{2}$ and $\mathbf{P}(\neg B|K_H) = 1$. Using Bayes rule:

$$\begin{aligned}\mathbf{P}(K_H|\neg B) &= \frac{\mathbf{P}(K_H) \times \mathbf{P}(\neg B|K_H)}{\mathbf{P}(\neg B)} \\ &= \frac{\mathbf{P}(K_H) \times \mathbf{P}(\neg B|K_H)}{\mathbf{P}(K_H) \times \mathbf{P}(\neg B|K_H) + \mathbf{P}(K_M) \times \mathbf{P}(\neg B|K_M)} \\ &= \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}} = \frac{2}{3}\end{aligned}$$

c) Since $\mathbf{P}(K_H)\times\mathbf{P}(\neg B|K_H) > \mathbf{P}(K_M)\times\mathbf{P}(\neg B|K_M)$, the map hypothesis is we chose the harmless kitty.

d) Let F_2 be the event that we feed our kitten 2 treats and do not get bitten. Let F_3 be the event that we feed our kitten the 3rd treat and do get bitten.

We have the probability we feed our kitten 2 treats and are not bitten or bitten as, $\mathbf{P}(F_2|K_M) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ and $\mathbf{P}(F_2|K_H) = 1$. Using the Law of total probability:

$$\mathbf{P}(F_2) = \mathbf{P}(K_H) \times \mathbf{P}(F_2|K_H) + \mathbf{P}(K_M) \times \mathbf{P}(F_2|K_M) = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{4} = \frac{5}{8}$$

Using Bayes rule:

$$\begin{aligned}\mathbf{P}(K_H|F_2) &= \frac{\mathbf{P}(K_H) \times \mathbf{P}(F_2|K_H)}{\mathbf{P}(F_2)} = \frac{\frac{1}{2} \times 1}{\frac{5}{8}} = \frac{4}{5} \\ \mathbf{P}(K_M|, F_2) &= \frac{\mathbf{P}(K_M) \times \mathbf{P}(F_2|K_M)}{\mathbf{P}(F_2)} = \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{5}{8}} = \frac{1}{5}\end{aligned}$$

And finally we solve using the Law of total probability:

$$\begin{aligned}
\mathbf{P}(F_3|F_2) &= \mathbf{P}(K_H|F_2) \times \mathbf{P}(F_3|K_H, F_2) + \mathbf{P}(K_M|F_2) \times \mathbf{P}(F_3|K_M, F_2) \\
&= \mathbf{P}(K_H|F_2) \times \mathbf{P}(F_3|K_H) + \mathbf{P}(K_M|F_2) \times \mathbf{P}(F_3|K_M) \\
&= \left(\frac{4}{5} \times 1\right) + \left(\frac{1}{5} \times \frac{1}{2}\right) = \frac{1}{10}
\end{aligned}$$

Problem 10 (14 points)

Consider two identical looking coins. One is fair so the probability of a head or tail resulting from a toss is equally likely. The second coin is biased with the probability of a toss resulting in a head is denoted by p and the probability of a toss resulting in a tail is $1 - p$.

A sequence of heads and tails is generated as follows. The "current coin" is tossed. If the result is a tail, we switch to the other coin for the next toss. If the result is a head, we toss the same coin to generate the next result. Define a discrete time Markov Chain where the state corresponds to the current coin, i.e. state 1 = biased, and state 2 = fair.

a) Draw the Markov Chain.

b) Write the transition probability matrix for this MC.

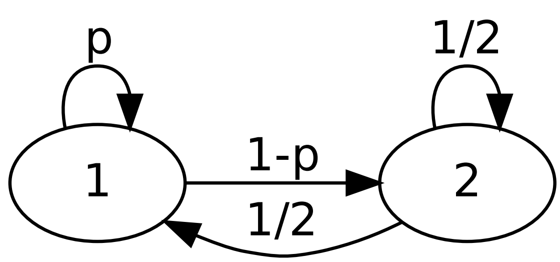
c) What is the solution for the stationary state probabilities?

d) Based on the result from part b, what is the long range fraction of coin tosses that result in heads?

e) Given that the result of the last toss was a head, what is the probability that the next toss result will be a head? You may leave your answer in terms of p, π_1, π_2, and arithmetic operations such as addition and multiplication.

Solution

a)



b)

$$\begin{pmatrix} p & 1-p \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

c)

Initial Equations	Solve for π_2 in terms of π_1	Plug into $1 = \pi_1 + \pi_2$	Solution
$\pi_1 = \pi_1 p + \frac{\pi_2}{2}$	$\frac{\pi_2}{2} = \pi_1(1-p)$	$1 = \pi_1 + \pi_1(2-2p)$	$\pi_1 = \frac{1}{3-2p}$
$\pi_2 = \pi_1(1-p) + \frac{\pi_2}{2}$	$\pi_2 = \pi_1(2-2p)$	$1 = \pi_1 + \pi_1(2-2p)$	$\pi_2 = \frac{2-2p}{3-2p}$
$1 = \pi_1 + \pi_2$			

d) $\pi_1 p + \frac{\pi_2}{2}$

e) Define H_t as the event we get heads at time t for any t. Consider some time t after the probabilities have converged. We want $P(H_{t+1}|H_t)$.

By the definition of conditional probability:

$$P(H_{t+1}|H_t) = \frac{P(H_{t+1} \cap H_t)}{P(H_t)}$$

The denominator we know from part d.

We can calculate the numerator by observing that:

$$\begin{aligned}
P\left(H_{t+1} \cap H_t\right) &= P(X_{t+2} = 1 \cap X_{t+1} = 1 \cap X_t = 1) + P(X_{t+2} = 2 \cap X_{t+1} = 2 \cap X_t = 2) \\
&= P(X_t = 1)P(X_{t+1} = 1|X_t = 1)P(X_{t+2} = 1|X_{t+1} = 1 \cap X_t = 1) \\
&\quad + P(X_t = 2)P(X_{t+1} = 2|X_t = 2)P(X_{t+2} = 2|X_{t+1} = 2 \cap X_t = 2) \\
&= P(X_t = 1)P(X_{t+1} = 1|X_t = 1)P(X_{t+2} = 1|X_{t+1} = 1) \\
&\quad + P(X_t = 2)P(X_{t+1} = 2|X_t = 2)P(X_{t+2} = 2|X_{t+1} = 2) \\
&= \pi_1 p^2 + \pi_2(1/4)
\end{aligned}$$

The second line follows from the multiplication rule, and the third line follows from the Markov assumption. Putting these together, we get

$$P(H_{t+1}|H_t) = \frac{\pi_1 p^2 + \pi_2(1/4)}{\pi_1 p + \pi_2/2}$$