LAST Name: SOLUTION

FIRST Name:

University ID:

Discussion Section:

PHYSICS 1C: Electrodynamics, Optics, and Special Relativity

Fall 2017, Lecture Series 2

Midterm 2—Thursday 16th November (VERSION A)

Time allowed: 50 minutes

Answer **all** *questions.*

Calculators **are** *permitted in this exam*

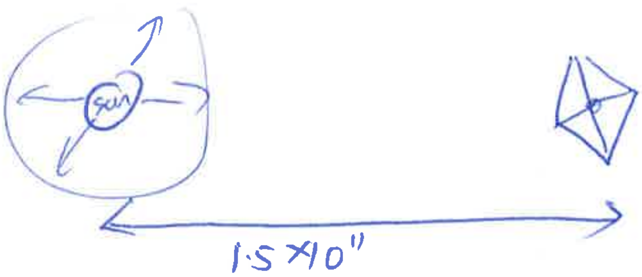
The numbers in the margin indicate the weight that the examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Question	Points
1	/12
2	/12
3	/25
4	/25
5	/26

Section A

1. The IKAROS spacecraft has a solar sail with dimensions 14 m×14 m. Assuming the solar sail is totally reflecting, estimate the maximum force it experiences due to radiation pressure from the Sun at the radius of the Earth's orbit.
[The luminosity of the Sun, equal to the total power it emits as radiation, is approximately 3.8×10^{26} W. The radius of the Earth's orbit is approximately 1.5×10^{11} m. The average radiation pressure on a totally reflecting surface is given by $p_{\mathrm{rad}} = 2I/c$, where the symbols have their usual meaning]. [12]



Intensity at Earth's orbit: $I = \dfrac{L}{4\pi R^2}$

$$p_{rad} = 2I/c$$

$$F_{rad} = p_{rad} \times A$$

$$F_{rad} = \frac{2LA}{4\pi R^2 c} = \frac{2(3.8\times10^{26})(14\times14)}{4\pi(1.5\times10^{11})^2(3\times10^8)}$$

$$= 1.75612\ldots\times10^{-3}\ \mathrm{N}$$

$$= 1.76\times10^{-3}\ \mathrm{N}\ (3\ \mathrm{s.f.})$$

2.

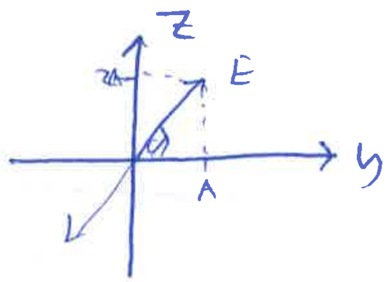
(a) Describe the polarisation state of the plane electromagnetic wave represented by the following electric field equations (where $A > 0$):

$$E_x = 0, \qquad E_y = A\cos(kx - \omega t), \qquad E_z = 2A\cos(kx - \omega t).$$

[Your description should include both the type of polarisation and any relevant details such as direction, etc.] oscillation [4]

(b) Determine the direction ~~and amplitude~~ of polarisation of the corresponding magnetic field. [8]

(a)



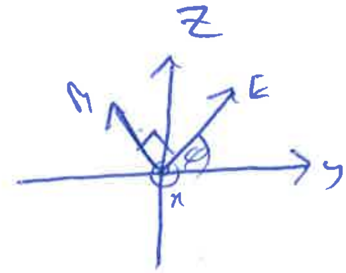
linearly polarised in yz-plane at ϕ to y-axis

$$\phi = \tan^{-1}\left(\frac{2A}{1A}\right) = 63.4349\ldots$$
$$= 63.4^\circ \ (3\text{s.f.})$$

(b) $\vec{E}\cdot\vec{B} = 0 \Rightarrow B_y E_y + B_z E_z = 0$

$$\Rightarrow B_y A + 2B_z A = 0$$

$$\Rightarrow B_y = -2B_z$$



$\vec{E}\times\vec{B} \propto \hat{c} \Rightarrow (E_y B_z - E_z B_y) = (AB_z - 2AB_y) > 0$

$\Rightarrow B_z < 0, \ B_y > 0$ <- These signs should be the other way round

$$\therefore \vec{B} = \begin{pmatrix} 0 \\ +2B_z \\ -B_z \end{pmatrix}$$

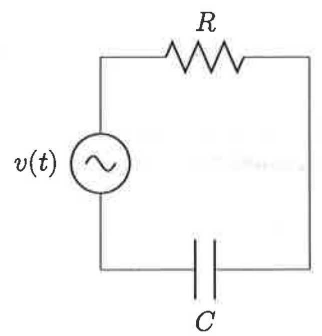
^These signs should be the other way round if B>0

linearly polarised in yz-plane at angle $90+\phi$ to y axis

$$90 + \phi = 153.4^\circ \ (3\text{s.f.})$$

Section B

3. A load consisting of a resistor R in series with a capacitor C is connected to a sinusoidal voltage supply with amplitude V_0 and angular frequency ω.

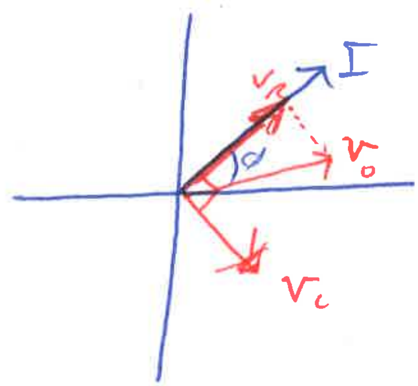


(a) With the aid of a phasor diagram, or otherwise, obtain the impedance and phase angle of the circuit in terms of ω, R and C. ~~What is the maximum energy stored in the capacitor during the cycle?~~ [14]

Assume $i(t) = I\cos(\omega t)$

$$\Rightarrow \quad V_R(t) = IR\cos(\omega t)$$

$$V_C(t) = \frac{I}{\omega C}\cos(\omega t - 90)$$



$$V_0 = \sqrt{V_R^2 + V_C^2} = I\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$\phi = -\tan^{-1}\left(\frac{V_C}{V_R}\right) = -\tan^{-1}\left(\frac{1/\omega C}{R}\right) = -\tan^{-1}\left(\frac{1}{\omega CR}\right)$$

(negative phase angle)

$$Z = V_0/I = \sqrt{R^2 + (1/\omega C)^2}$$

(b) Calculate the average power dissipated in the resistor in terms of V_0, ω, R and C. What is its value in the limits when i) $\omega CR \gg 1$ and ii) $\omega CR \ll 1$? [11]

$P_{AV} = \frac{1}{2} I V \cos\phi$

Resistor: $P_{AV} = \frac{1}{2} I (IR) = \frac{1}{2} I^2 R$

$$= \frac{1}{2}\left(\frac{V_0}{Z}\right)^2 R = \frac{V_0^2 R}{2\left(R^2 + \left(\frac{1}{\omega C}\right)^2\right)}$$

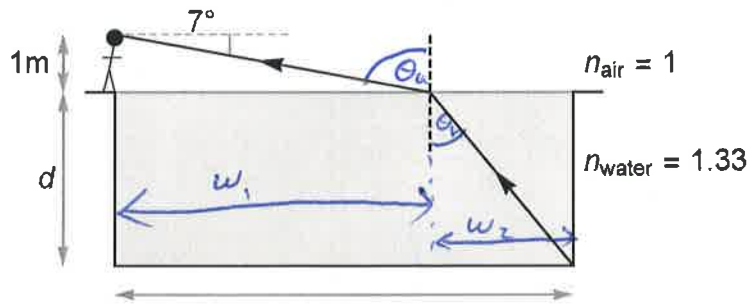
(i) $\omega CR \gg 1 \Rightarrow R \gg 1/\omega C$

$$P_{AV} \longrightarrow \frac{V_0^2 R}{2(R^2)} = \frac{V_0^2}{2R}$$

(ii) $\omega CR \ll 1 \Rightarrow R \ll 1/\omega C$

$$P_A \longrightarrow \frac{V_0^2 R}{2(1/\omega C)^2} = \frac{V_0^2 \omega^2 C^2 R}{2}$$

4. Ryan is calculating the depth of a rectangular pool full of water. He has measured its length as 10m, and notices that if he looks 7° below the horizontal he can see the bottom corner of the far side of the pool, as shown in the diagram below (ignoring reflected rays).



(a) Assuming Ryan's eyes are 1m above the ground, and that the refractive indices of air and water are $n_{air} = 1$ and $n_{water} = 1.33$, calculate the depth d of the pool. [15] ~~[13]~~

$$\theta_a = 90 - 7 = 83°$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}}$$

$$n_a \sin\theta_a = n_b \sin\theta_b \quad \Rightarrow \quad \theta_b = \sin^{-1}\left(\frac{n_a}{n_b}\sin\theta_a\right)$$

$$w_1 = \tan\theta_a = 8.144\ldots \text{m}$$

$$w_2 = 10 - w_1$$

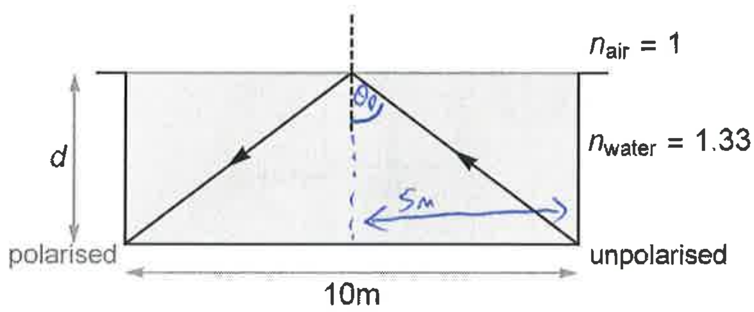
$$d = \frac{w_2}{\tan\theta_b} = \frac{10 - w_1}{\tan\left[\sin^{-1}\left(\frac{n_a}{n_b}\sin\theta_a\right)\right]}$$

$$= \frac{10 - 8.144\ldots}{\tan\left[\sin^{-1}\left(\frac{1}{1.33}\sin(83)\right)\right]}$$

$$= 1.655143\ldots$$

$$= \underline{1.66 \text{ m (3 s.f.)}}$$

(b) In a different pool, also 10 m in length, Ryan finds that a beam of unpolarised light shone from the bottom corner to the centre of the surface is fully polarised after reflection, as shown in the diagram below (which ignores refracted rays).



In which direction is the reflected light polarised? What is the depth d of this second pool? ~~[12]~~ [10]

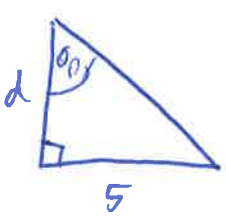
Light is polarised perpendicular to the plane of incidence [i.e. into / out of the page].

Beam must strike surface at θ_p:

$$\theta_p = \tan^{-1}\left(\frac{n_b}{n_a}\right) = \tan^{-1}\left(\frac{1}{1.33}\right)$$

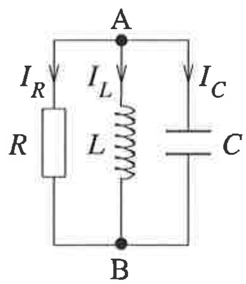
$$= 36.938...^\circ$$

$$= 36.9^\circ \text{ (3sf.)}$$



$$d = \frac{5}{\tan \theta_p} = 6.65\text{m}$$

5. In the parallel resonant circuit below, the voltage across points A and B is V.



The current through the capacitor can be written in terms of V and C using the equation $V = q/C$, so that

$$I_C = \frac{\mathrm{d}q}{\mathrm{d}t} = C\frac{\mathrm{d}V}{\mathrm{d}t}.$$

(a) Obtain similar expressions for the currents I_R and I_L in terms of V, R and L. By using Kirchhoff's circuit law for currents, show that the voltage V satisfies the equation

$$C\frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}V}{\mathrm{d}t} + \beta V = 0, \qquad (\star)$$

where α and β are constants (which you should find). [14]

R: $V = I_R R \Rightarrow I_R = V/R$

L: $V = L\frac{dI}{dt} \Rightarrow \frac{dI_L}{dt} = \frac{V}{L} \Rightarrow I_L = \frac{1}{L}\int V\,dt$

K_1: $I_R + I_L + I_C = 0$

$$\Rightarrow \frac{V}{R} + \frac{1}{L}\int V\,dt + C\frac{dV}{dt} = 0$$

Differentiate with respect to time:

$$\frac{1}{R}\frac{dV}{dt} + \frac{1}{L}V + C\frac{d^2V}{dt^2} = 0$$

$$\left[\begin{array}{c} \alpha = 1/R \\ \beta = 1/L \end{array}\right]$$

(b) Show by substitution, or otherwise, that a solution to this differential equation is given by

$$V(t) \quad = \quad Ae^{-\gamma t}\cos(\omega t),$$

and find expressions for γ and ω in terms of R, L and C.

[You may assume that $4R^2 > L/C$. If you did not complete part (a), you may leave your answer in terms of α and β for full credit. Hint: for the left hand side of Equation ($\star$) to be zero, terms proportional to $\sin(\omega t)$ and terms proportional to $\cos(\omega t)$ should sum to zero independently.]. [12]

$$V(t) = Ae^{-\gamma t}\cos(\omega t)$$

$$\frac{dV}{dt} = A(-\gamma e^{-\gamma t})\cos(\omega t) + Ae^{-\gamma t}(-\omega\sin(\omega t))$$

$$= Ae^{-\gamma t}\left[-\gamma\cos(\omega t) - \omega\sin(\omega t)\right]$$

$$\frac{d^2V}{dt^2} = A(-\gamma)e^{-\gamma t}\left[-\gamma\cos(\omega t) - \omega\sin(\omega t)\right] + Ae^{-\gamma t}\left[+\gamma\omega\sin(\omega t) - \omega^2\cos(\omega t)\right]$$

$$= Ae^{-\gamma t}\left[(\gamma^2 - \omega^2)\cos(\omega t) + (2\gamma\omega)\sin(\omega t)\right]$$

$\Leftrightarrow (*)$

$$\frac{1}{R} A e^{-\gamma t}\left[-\gamma\cos(\omega t) - \omega\sin(\omega t)\right]$$

$$+\frac{1}{L} A e^{-\gamma t}\left[\cos(\omega t)\right]$$

$$+ C A e^{-\gamma t}\left[(\gamma^2-\omega^2)\cos(\omega t) + (2\gamma\omega)\sin(\omega t)\right] = 0$$

$$\Rightarrow A e^{-\gamma t}\left[\cos(\omega t)\left[\overbrace{-\frac{\gamma}{R}+\frac{1}{L}+(\gamma^2-\omega^2)C}^{①}\right] + \sin(\omega t)\left[\underbrace{-\frac{\omega}{R}+2\gamma\omega C}_{②}\right]\right] = 0$$

$$② \Rightarrow -\frac{\omega}{R} + 2\gamma\omega C = 0$$

$$\Rightarrow 2\gamma C = \frac{1}{R}$$

$$\Rightarrow \gamma = \frac{1}{2RC}$$

$$① \Rightarrow -\frac{1}{R}\left(\frac{1}{2RC}\right) + \frac{1}{L} + \left(\frac{1}{4R^2C^2} - \omega^2\right)C = 0$$

$$\Rightarrow -\frac{1}{2R^2C} + \frac{1}{L} + \frac{1}{4R^2C} = \omega^2 C$$

$$\Rightarrow \omega^2 = \frac{1}{LC} - \frac{1}{4R^2C^2} \Rightarrow \omega = (\pm)\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$