LAST Name: SOLUTION

FIRST Name:

University ID:

PHYSICS 1C: Electrodynamics, Optics, and Special Relativity

Fall 2017, Lecture Series 2

Final Exam—Tuesday 12th December 11:30am

Time allowed: 3 hours

Answer **all** *questions.*

Calculators **are** *permitted in this exam*

The numbers in the margin indicate the weight that the examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Question	Points
1	/10
2	/6
3	/8
4	/14
~~5~~	~~/12~~
6	/16
7	/18
8	/18
9	/18
Total	/120

2x

24 6
143 1

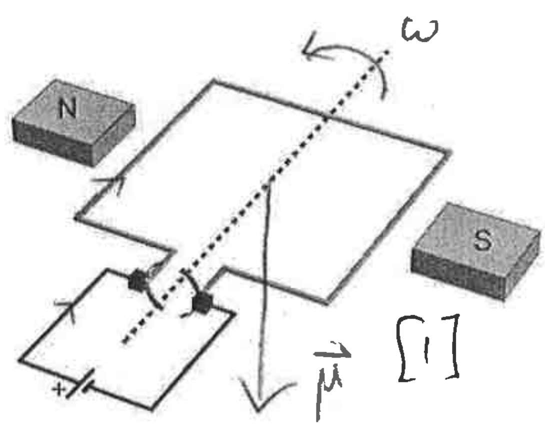
14

36x3

Section A

1.

(a) A simple DC motor consists of a rotating wire loop (rotor), which is connected to a DC voltage supply via slip rings and brush contacts and placed in a static magnetic field, as shown in the figure below. Draw on the diagram the direction of the magnetic dipole moment of the rotor and the direction of its rotation.



Does the DC motor experience a back-torque or a back-emf? [2]

Back-emf [1]

(b) The lens focal length f, aperture diameter D, and shutter time T (i.e. the time for which light is let into the camera) for four different cameras are given below. Which has the largest exposure (i.e. the greatest amount of light energy per unit area reaching the film)?

AC BD

0.04×10^{-3} (i) A camera with $f = 50$ mm, $D = 10$ mm, $T = 10$ ms. ☐ ☒

0.0625×10^{-3} (ii) A camera with $f = 80$ mm, $D = 20$ mm, $T = 10$ ms. ☒ ☐ [2]

0.055×10^{-3} (iii) A camera with $f = 60$ mm, $D = 10$ mm, $T = 20$ ms. ☐ ☐

0.0125×10^{-3} (iv) A camera with $f = 80$ mm, $D = 4$ mm, $T = 50$ ms. ☐ ☐ for correct answer

[2]

Exposure $\propto \dfrac{D^2}{f^2} T$ [1] for this if answer incorrect.

5.45×10^{-19} [BD]

 [A C]

(c) A proton has rest mass $m_p = 1.67 \times 10^{-27}$ kg and relativistic momentum 3.76×10^{-19} kg m/s. What is its total energy? [3]

$$E^2 - p^2c^2 = m^2c^4 \quad [1]$$

$$\Rightarrow E = \sqrt{(3.76\times10^{-19})^2(3\times10^8)^2 + (1.67\times10^{-27})^2(3\times10^8)^4}$$

$$= 1.8792\ldots \times 10^{-10}\ \mathrm{J}$$

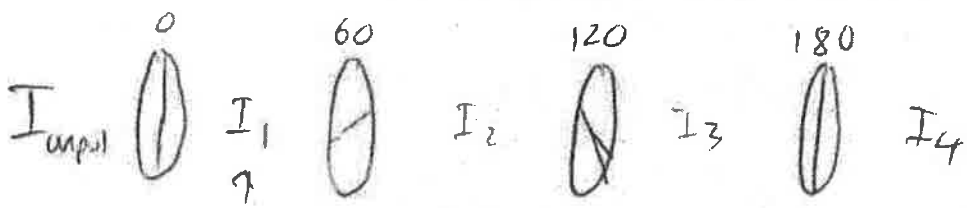
$$= \underline{\underline{1.88 \times 10^{-10}\ \mathrm{J}}} \quad (3\,\mathrm{s.f.})$$

[2]

2.22×10^{-10} J [BD]

(full marks for correct answer)

(d) Unpolarised light with intensity I passes through a stack of four polarisers, each of which has its polarisation axis aligned at 60° with respect to that of the previous one. What is the final intensity of light that passes through the stack? [3]



[1] (Malus's law)

$I_{pol} = \frac{1}{2} I_{unpol}$ [1]

$$I_4 = I_3 \cos^2(60) = I_2 \cos^4(60) = I_1 \cos^6(60)$$

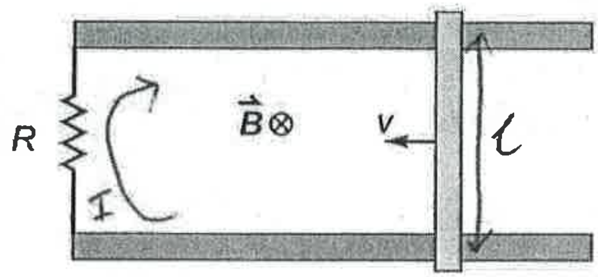
$$= \frac{I \cos^6(60)}{2} = I\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^6 = \underline{\underline{\frac{I}{128}}} \quad [1]$$

[full marks for C.A.]

BD: 4.0ms^{-1}

AC

2. A 0.50 m metal bar is pulled to the left at a steady speed of 2.0 m/s perpendicular to a uniform magnetic field with strength $B = 1.0$ T. The bar rides on parallel metal rails connected through a 100 Ω resistor so that a complete circuit is formed, as shown in the figure below.



[1]

Calculate the induced current through the resistor, and indicate its direction on the figure above. [Ignore the resistance of the bar and the rails]. [6]

$$dA = -v\,dt\,l$$

$$d\Phi = B\,dA = -vB\,dt\,l$$

$$\Rightarrow \frac{d\Phi}{dt} = -vBl \quad [2]$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = vBl \quad \cancel{[2]}\ [1]$$

$$I = \frac{\mathcal{E}}{R} = \frac{vBl}{R} = \frac{(2)(1)(0.5)}{100}$$

$$= \underline{\underline{0.01\text{ A}}} \quad [2]$$

AC

BD:

$\underline{\underline{0.02\text{ A}}}$

full marks for C.A.

3. An electromagnetic wave travelling in a vacuum has an electric field given by

$$\vec{E}(y,t) = (3.00 \times 10^5\ \text{V/m})\,\hat{k}\cos\left[ky - (12 \times 10^{12}\ \text{rad/s})\,t\right].$$

(a) In which direction is the wave travelling? [1]

(b) What is the wavelength λ of the wave? [3]

(c) Obtain an expression for the Poynting vector of the wave $\vec{S}$, including units in your answer (not k, ω?) [4]

(a) $ky - \omega t \Rightarrow$ positive y-direction [1]

(b) $\lambda = \frac{2\pi}{k}$ $\quad \omega = ck \Rightarrow \lambda = \frac{2\pi c}{\omega}$ [2]

$$= \frac{2\pi(3\times10^8)}{12\times10^{12}} = 1.5707\ldots \times 10^{-4}\,\text{m}$$

$$= 1.57\times10^{-4}\,\text{m (3s.f.)}$$ [1]

(c) $E_{max} = cB_{max} \Rightarrow B = \frac{3\times10^5}{3\times10^8} = 10^{-3}\,\text{T}$ (full marks for CA)

$$\vec{B}(y,t) = (10^{-3}\,\text{T})\,\hat{i}\cos[ky - \omega t]$$ [1]

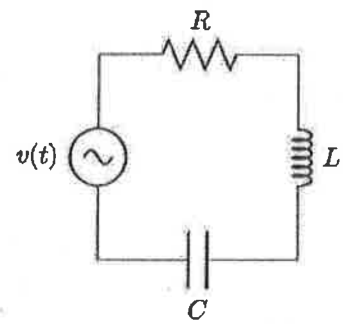
$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{(3\times10^2)}{\mu_0}\hat{j}\cos^2(ky-\omega t)$$

$$= (2.3873\ldots\times10^8\ \text{J/m}^2/\text{s})\,\hat{j}\cos^2(ky-\omega t)$$

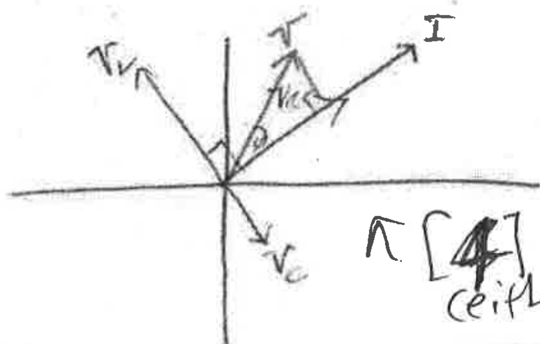
$$= \underline{(2.39\times10^8\ \text{J}\,\text{m}^{-2}\,\text{s}^{-1})\,\hat{j}\cos^2(ky-\omega t)}$$

[1] (magnitude) [1] (units) [1] (rest)

4. An L-R-C series circuit is connected to an AC source of constant voltage amplitude V and variable angular frequency ω, as shown in the figure below.



(a) Using a phasor diagram, or otherwise, obtain an expression for I, the amplitude of the current through the circuit in terms of L, R, C, ω and V. + phase angle. [6]



$V_R(t) = IR\cos(\omega t)$

$V_C(t) = \frac{I}{\omega C}\cos(\omega t - 90°)$

$V_L(t) = I\omega L\cos(\omega t + 90°)$

[4] (either)

$$V = \sqrt{R^2 + (\omega L - 1/\omega C)^2}\ I \quad [1]$$

$$\Rightarrow I = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad [1]$$

(full marks for correct answer)

$$\phi = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right) = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

emph

(b) Hence find an expression for P_R, the average power dissipated in the resistor in terms of L, R, C, ω and V. At what value of ω is P_R a maximum? ~~Give a rough~~ sketch of P_R as a function of ω. *Label coordinates of maximum [8]

$$P_R = \frac{1}{2} I V \cos\phi \quad [1] \quad (\phi_R = 0)$$

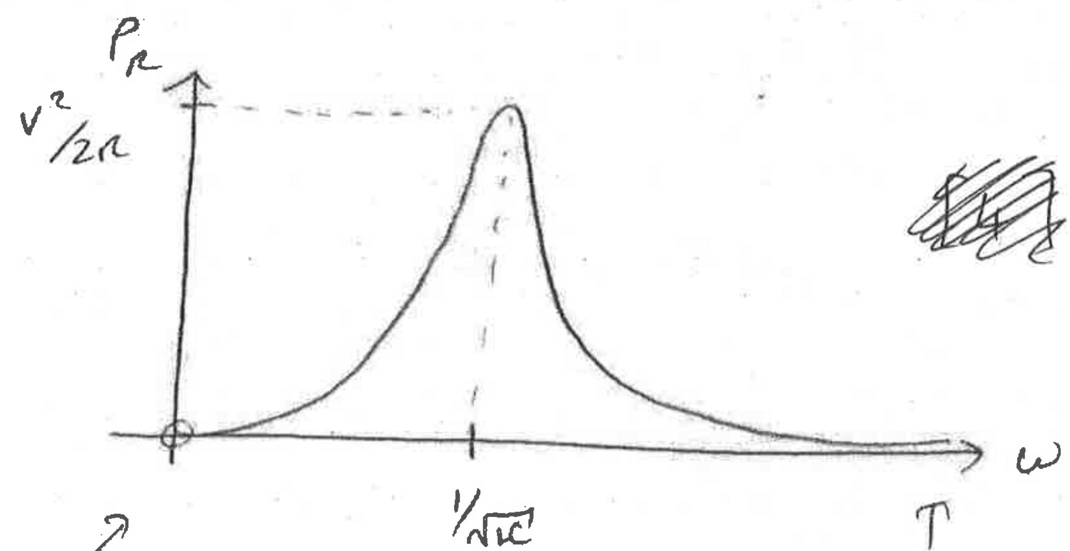
current through R — voltage across R

$$= \frac{1}{2} I^2 R = \frac{1}{2} \frac{V^2 R}{\left(R^2 + (\omega L - 1/\omega C)^2\right)} \quad [3]$$

P_R max when denominator smallest.

$$\Rightarrow \quad \omega L - 1/\omega C = 0$$

$$\Rightarrow \quad \omega^2 = 1/LC \Rightarrow \omega = \frac{1}{\sqrt{LC}} \quad [1]$$



[3]

$P(\omega \to 0) \to 0$

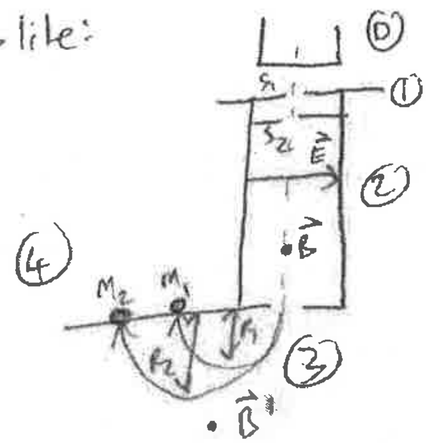
$P(\omega \to \infty) \to 0.$

+ More space:

5. Explain, with the aid of diagrams and relevant equations, how a mass spectrometer works. [12]

A mass spectrometer measures the masses of ions using electric and magnetic fields.

The setup looks like:



⓪ At stage 0, a sample is ionised (e.g. by electron bombardment)

① At ①, positively charged ions are accelerated and focussed through slits.

② At ②, the positively charged ions are exposed to crossed electric and magnetic fields, experiencing a Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Particles are deflected unless $\vec{F} = 0 \Rightarrow |v| = |E|/|B|$

③ Particles exiting all have the same velocity and perform cyclotron motion in magnetic field $\vec{B}'$:

$$F = \frac{mv^2}{r} = qvB' \Rightarrow r = \frac{mv}{qB'} = \frac{mE}{qBB'}$$

④ Since $r \propto m$, different masses may be identified on a detector.

BD: 4.00mm

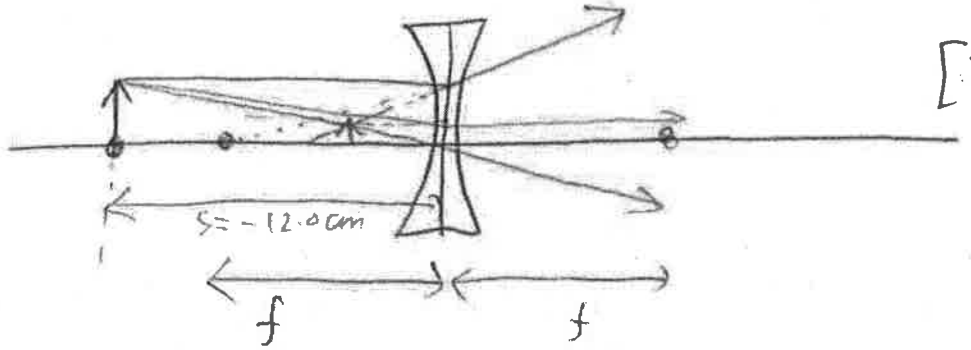
Section B

6.

(a) A thin lens with focal length of -12.0 cm is placed 18.0 cm to the right of an object that is 3.00 mm tall. Draw a principal ray diagram of this setup and determine the location, size, nature (real or virtual) and orientation (inverted or upright/erect) of the resulting image I_1.

AC

[7]



[3] (only two rays need to be drawn)

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow s' = \frac{1}{\frac{1}{f} - \frac{1}{s}}$$

$$= \frac{1}{(-12\times10^{-2})^{-1} - (18\times10^{-2})^{-1}} = -7.2\text{cm}$$

$$m = -\frac{s'}{s} = \frac{7.2}{18.0} = 0.4$$

[2]

1) Location: $s' = -7.2$cm (to left of lens)

2) Size $y' = my = 0.4 \times 3.00 = 1.2$mm [1]

AC

3) Virtual

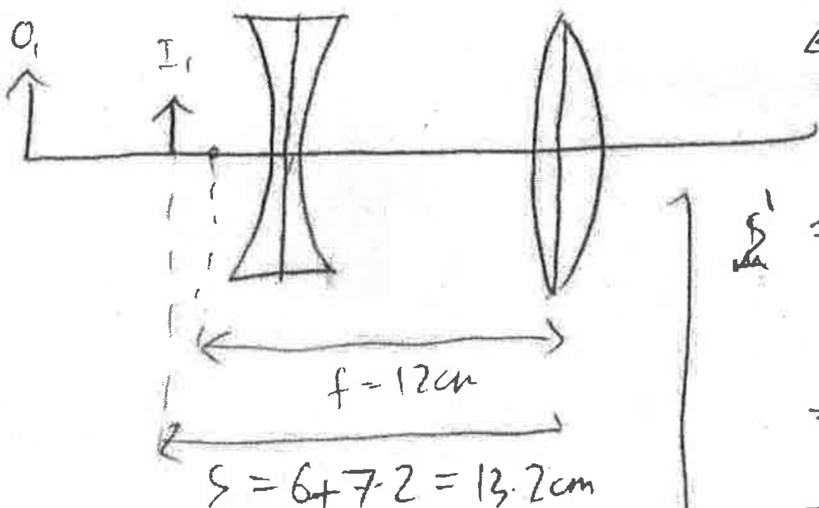
4) Upright [1]

BD:

1.6mm

(b) A second thin lens with focal length +12.0 cm is now placed 6.00 cm to the right of the first lens. What is the location, size, nature and orientation of the final image I_2? What is the overall lateral magnification of the combination of lenses? [9]

I_1 acts as object for lens 2 ← [1] (may be implicit)



← diagram optional here.

$$s' = \frac{1}{\frac{1}{f} - \frac{1}{s}} = \frac{1}{(12\times10^{-1})^{-1} - (13.2\times10^{-2})^{-1}} = 1.32\,\text{m} \leftarrow [3]$$

$$m_2 = -\frac{s'}{s} = -\frac{1.32}{13.2\times10^{-2}} = 10$$

1) Location: 1.32 m to right of lens 2 [1]

2) Size: $y_2' = m y_1 = 10 \times 1.2\,\text{mm} = 12\,\text{mm}$ [1]

AC | BD: 16 mm

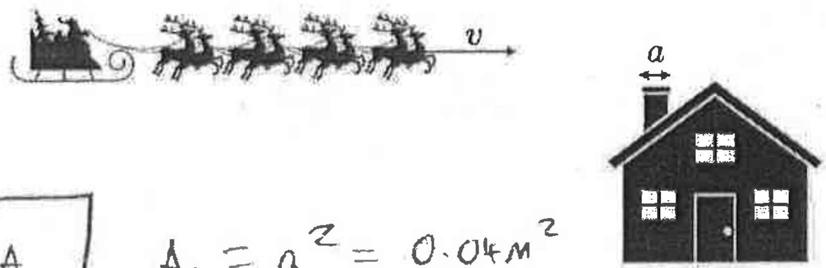
3) Real

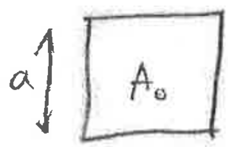
4) Inverted [1]

Total magnification: $m = m_1 m_2 = 0.4 \times 10 = 4$ [2]

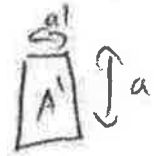
7. Santa Claus is coming to town! In order to maximise the gift delivery rate, Santa's sleigh is pulled by *relativistic reindeer*, capable of travelling at speeds close to the speed of light.

(a) A particular chimney has a square cross-sectional area with side length $a = 20$ cm in its rest frame. What is the area of this chimney as observed by Santa, who is travelling horizontally at a speed of $v = (3/5)c$ relative to the chimney, perpendicular to one of its sides? [4]





$A_0 = a^2 = 0.04\,m^2$



To Santa, one ~~[illegible]~~ dimension is contracted:

[1] → $a' = \frac{a}{\gamma} = a \times \sqrt{1-\left(\frac{3}{5}\right)^2}$ [1] $= a \times \sqrt{\frac{16}{25}}$

$= \frac{4a}{5} = \frac{4}{5} \times (20 \times 10^{-2})$

$= 16$ cm [1]

$\therefore A' = a \times a' = \underline{0.032\,m^2}$ [1]

(b) One of Santa's reindeer, Rudolph, has a very shiny nose which emits light radiation in the red part of the visible spectrum with wavelength $\lambda_{\text{red}} = 700$ nm (in its rest frame). According to an observer on earth, standing in the path of the oncoming reindeer, Rudolph's nose appears to be emitting blue light with wavelength $\lambda_{\text{blue}} = 460$ nm. How fast are the reindeer travelling in this situation? [6]

$$f = \sqrt{\frac{c+u}{c-u}}\, f_0 \Rightarrow \lambda = \sqrt{\frac{c-u}{c+u}}\, \lambda_0 \quad [2]$$

$$\Rightarrow \lambda_b = \sqrt{\frac{c-u}{c+u}}\, \lambda_r = \left(\frac{\lambda_b}{\lambda_r}\right)^2 = \frac{c-u}{c+u}$$

$$\Rightarrow \left(\frac{\lambda_b}{\lambda_r}\right)^2 (c+u) = c - u$$

$$\Rightarrow u\left[\left(\frac{\lambda_b}{\lambda_r}\right)^2 + 1\right] = c\left[1 - \left(\frac{\lambda_b}{\lambda_r}\right)^2\right]$$

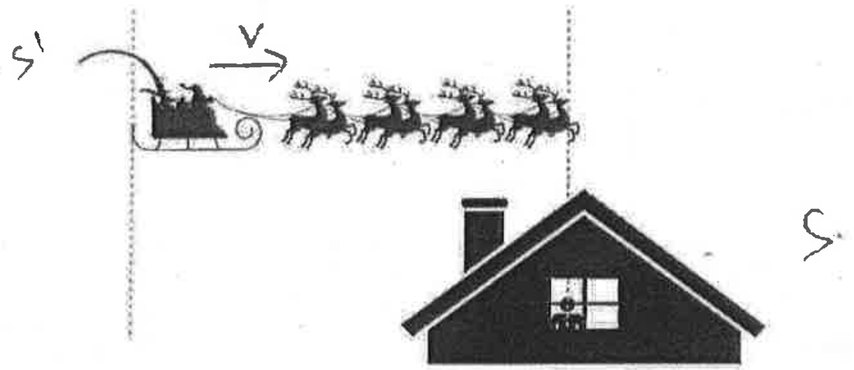
$$\Rightarrow u = c\left[\frac{1 - \left(\frac{\lambda_b}{\lambda_r}\right)^2}{1 + \left(\frac{\lambda_b}{\lambda_r}\right)^2}\right] \quad [2]$$

$$= c\left[\frac{1 - \left(\frac{460}{700}\right)^2}{1 + \left(\frac{460}{700}\right)^2}\right]$$

$$= c(0.3968\ldots)$$

$$= \underline{\underline{(0.397)c}} \quad [2]$$

(c) Santa's sleigh plus reindeer has a total length of 10 m, measured from the back of the sleigh to the front of Rudolph's nose (in the frame in which the sleigh is at rest). During one part of the journey, Santa is travelling at a speed of (3/5)c relative to the earth along the x-axis, and is being secretly observed by Ryan from his bedroom window. At the instant that Rudolph's nose is directly above Ryan (event 1), both Santa and Ryan's personal clocks show time $t = 0$. At the same instant, in Santa's frame of reference, a gift is dropped from the back of the sleigh (event 2). At what time is the gift dropped from Santa's sleigh according to Ryan? [8]



	S	S'	
E1:	$x_1 = 0$, $t_1 = 0$	$x_1' = 0$, $t_1' = 0$	[Rudolph's nose above Ryan]
E2:	$x_2 = ?$ (not needed), $t_2 = ?$	$x_2' = -10\,m$, $t_2' = 0$	[4] (may be implicit)

Lorentz: $t' = \gamma\left(t - \frac{ux}{c^2}\right) \Rightarrow t = \gamma\left(t' + \frac{ux'}{c^2}\right)$ [1]

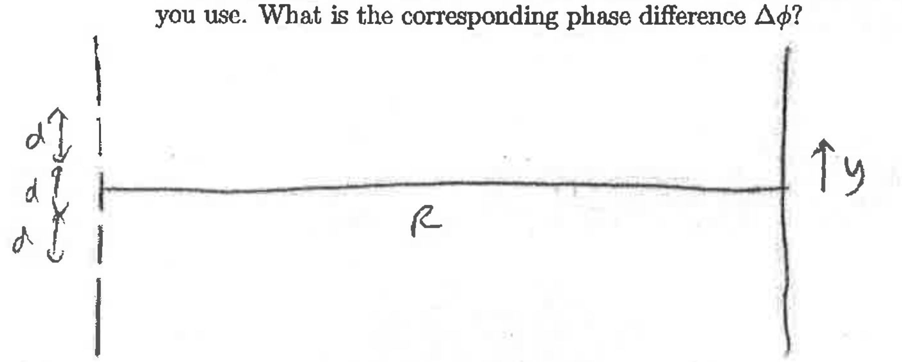
$$\Rightarrow t_2 = \frac{1}{\sqrt{1-(3/5)^2}}\left(0 + \frac{(\frac{3}{5}c)(-10)}{c^2}\right) \quad [1]$$

$$= \frac{5}{4}\left(-\frac{6}{c}\right) = -\frac{15}{2c} = \underline{\underline{-2.5\times10^{-8}\,s}} \quad [2]$$

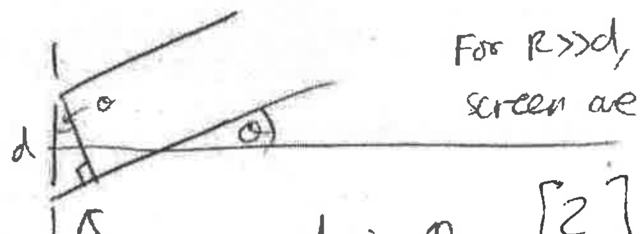
8. Four identical parallel narrow slits are cut into an opaque piece of card, with neighbouring slits separated by a distance d. Coherent, monochromatic light with wavelength λ is shone on the slits, and detected on a screen a distance $R \gg d$ away.

(a) What is the difference in path length travelled by light from neighbouring slits that reaches the screen at position y, where $y = 0$ corresponds to the centre of the screen? Draw a diagram to justify your answer, and explain any approximations you use. What is the corresponding phase difference $\Delta\phi$? [6]

λ



Two neighbouring slits:



For $R \gg d$, paths to screen are approximately parallel.

$\Delta r = d \sin\theta$ [2]

For $R \gg d$, $\sin\theta \approx \tan\theta = \frac{y}{R}$ [1] ($\sin\theta \approx \tan\theta$ should be mentioned)

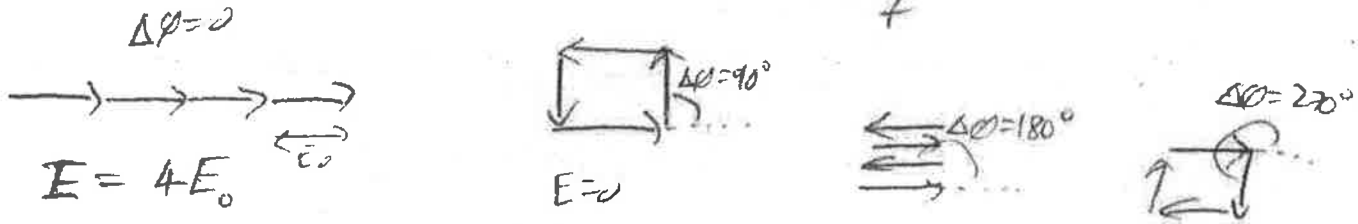
$\Rightarrow \quad \underline{\Delta r = \frac{dy}{R}}$ [1]

$\Delta\phi = \frac{2\pi}{\lambda}\Delta r = \underline{\frac{2\pi d y}{R\lambda}}$ [2]

Light reaching the screen from a given slit can be represented as a phasor. Phasors from neighbouring slits are offset by a relative phase difference of $\Delta\phi$.

(b) Show, with the help of phasor diagrams, that there are three values of $\Delta\phi$ in the range $0 \leq \Delta\phi < 2\pi$ for which the total intensity of light reaching the screen is zero. If the intensity of the incoming light is I_0, what is the maximum intensity observed on the screen? Draw a rough plot of the expected intensity pattern observed on the screen as a function of y around $y = 0$, showing at least ~~3~~ 7 principle maxima and labelling their coordinates. [Other coordinates do not need to be labelled]. [6]

+give these values.



$E = 4E_0$ — primary maximum

$E = 0$ — minimum

minimum

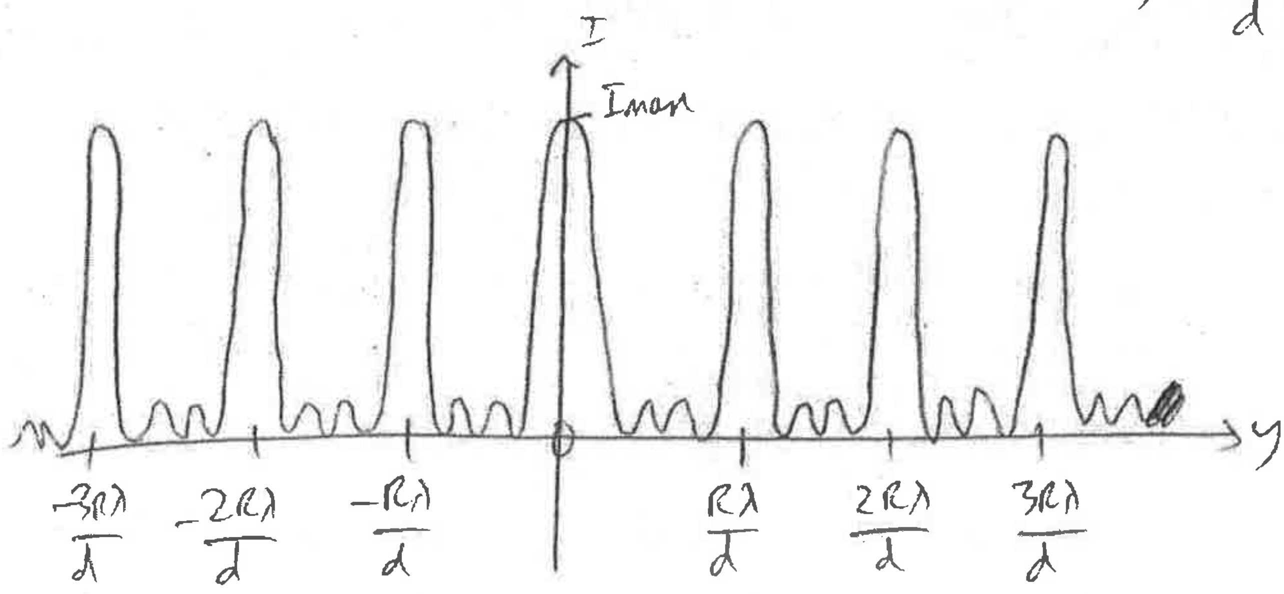
minimum

minimum when $\Delta\phi = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ [3]

$E_{max} = 4E_0 \Rightarrow I_{max} = 16 I_0$ [1]

$\Delta\phi = 2\pi m = \frac{2\pi d y}{R\lambda}$

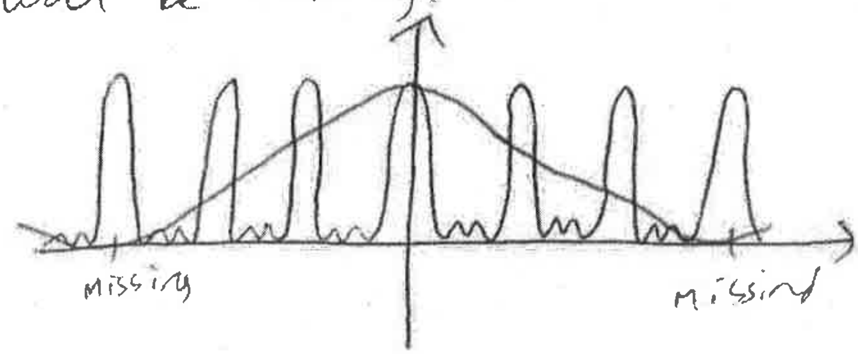
$\Rightarrow y = \frac{R\lambda m}{d}$



[2]

(c) You inspect the screen and note that the central principal maximum and two principal maxima each side of this are present as expected. However, the next principal maximum on each side of these five is missing, and the intensity at these points is zero. How can this be explained by the slits having a finite width? What is the finite width a of each slit, in terms of the distance d between them? [6]

A single slit with finite width has its own intensity pattern, which multiplies the intensity pattern of four ideal slits. Minima in the single-slit pattern may coincide with principal maxima from the 4-slit pattern, and so these will be missing.



[2]

4th principal maximum of 4 slits: $y_4 = \frac{3R\lambda}{d}$ [1]

1st minimum of finite-slit pattern: $\frac{\tilde{y}_*}{R} = \frac{m\lambda}{a}$ $(m = \pm 1, \pm 2, \ldots)$

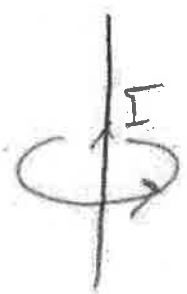
$\tilde{y}_1 = \frac{\lambda R}{a}$ [1]

For $\tilde{y}_1 = y_3 \Rightarrow$ $\underline{\underline{a = \frac{d}{3}}}$ [2]

9.

(a) State Ampère's law (in the absence of changing electric fields), and use it to find the magnitude B of the magnetic field a distance r from an infinite straight wire that carries a current I. Show clearly on a diagram the direction of the magnetic field with respect to that of the current. [Be sure to state any symmetry assumptions you make.] [5]

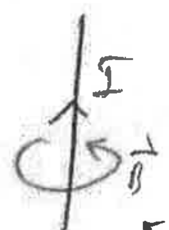
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} \quad [1]$$



By symmetry, magnetic field must be azimuthal around wire → so choose a circular contour. [1]

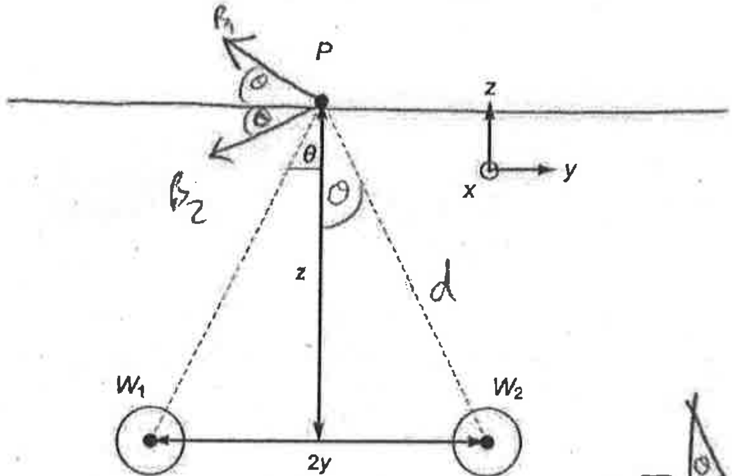
$$2\pi r B = \mu_0 I \Rightarrow B(r) = \frac{\mu_0 I}{2\pi r}$$

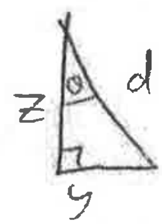
$$\Rightarrow \underline{\underline{\vec{B}(r) = \frac{\mu_0 I}{2\pi r} \hat{\theta}}} \quad [2]$$



[1] (direction)

(b) Two parallel wires W_1 and W_2, separated by a distance $2y$, each carry a current I in the x-direction. Using your answer from part (a), and the principle of superposition, find the magnitude and direction of the total magnetic field at the point P, a distance z above the midpoint between the two wires. [Hint: the angle labelled θ is the only angle you need for this problem. You can express $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of y and z using a right-angled triangle]. [8]





$d = \sqrt{y^2 + z^2}$ [1]

$\cos\theta = \frac{z}{d} = \frac{z}{\sqrt{y^2+z^2}}$ [1]

$|B_1| = \frac{\mu_0 I}{2\pi d} = \frac{\mu_0 I}{2\pi\sqrt{y^2+z^2}}$ [2]

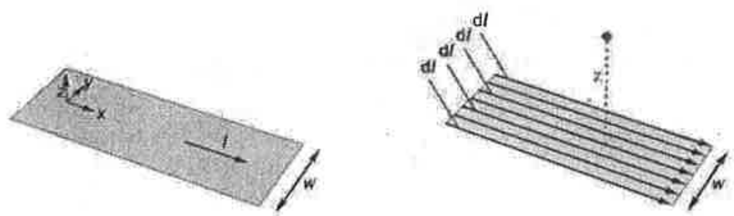
z-component cancels by symmetry, only y-component remains.

$\Rightarrow B_{tot} = 2|B|\cos\theta$ ← [2] (may be on diagram)

$= \frac{\mu_0 I}{\pi\sqrt{y^2+z^2}} \times \frac{z}{\sqrt{y^2+z^2}}$

$= \frac{\mu_0 I z}{\pi(y^2+z^2)} \Rightarrow \vec{B} = \frac{\mu_0 I z}{\pi(y^2+z^2)}(-\hat{y})$ [2]

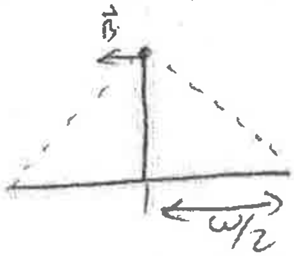
(c) An infinitely long conducting plate of width w and of negligible thickness lies in the xy-plane, with its infinite direction along the x-axis. It carries a uniform current I in the positive x-direction. The conducting plate can be modelled as many parallel infinite wires each carrying a small element of current $\mathrm{d}I = \lambda \mathrm{d}y$, where $\lambda = I/w$ is the current per unit length along the width of the plate:



Using your answer to part (b), or otherwise, show that the total magnetic field at a point a distance z vertically above the middle of the plate is

$$B = \frac{\mu_0 I}{\pi w} \tan^{-1}\left(C\frac{w}{z}\right),$$

where C is a numerical factor (which you should find). [Hint: you may need one of the integrals $\int_0^b \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left[\frac{b}{a}\right]$ or $\int_{-b}^{b} \frac{\mathrm{d}x}{a^2+x^2} = \frac{2}{a}\tan^{-1}\left[\frac{b}{a}\right]$.] [5]



From (b), field due to current elements at $\pm y$

$$dB = \frac{\mu_0 (\overbrace{\lambda dy}^{dI}) z}{\pi (z^2 + y^2)} \quad [2]$$

Integrate from $y=0$ to $y = w/2$:

$$B = \int dB = \underbrace{\frac{\lambda \mu_0 z}{\pi} \int_0^{w/2} \frac{dy}{z^2+y^2}}_{[1]} = \frac{\lambda \mu_0 z}{\pi}\left[\frac{1}{z}\tan^{-1}\left(\frac{w}{2z}\right)\right]$$

$$= \frac{\lambda \mu_0}{\pi}\tan^{-1}\left(\frac{w}{2z}\right) = \underline{\frac{\mu_0 I}{\pi w}\tan^{-1}\left(\frac{w}{2z}\right)} \quad [2]$$