83 + 2 = 85

Physics 1BH – Prof. J. Rosenzweig – Winter 2017
Midterm 1
February 2, 2017

Use only the paper provided for you. Show all of your work for full credit. Write your name on each sheet of paper in your answers, then staple all together in order. You have 1 hour and 50 minutes to complete this exam. You are permitted one sheet of paper as notes, with writing on both sides.

1. Consider a scalar potential that is a function of only x and y,

$$\phi(x,y) = V\exp\left[-\frac{(x^2+y^2)}{2a^2}\right],$$

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where V and a are constants with the appropriate units.

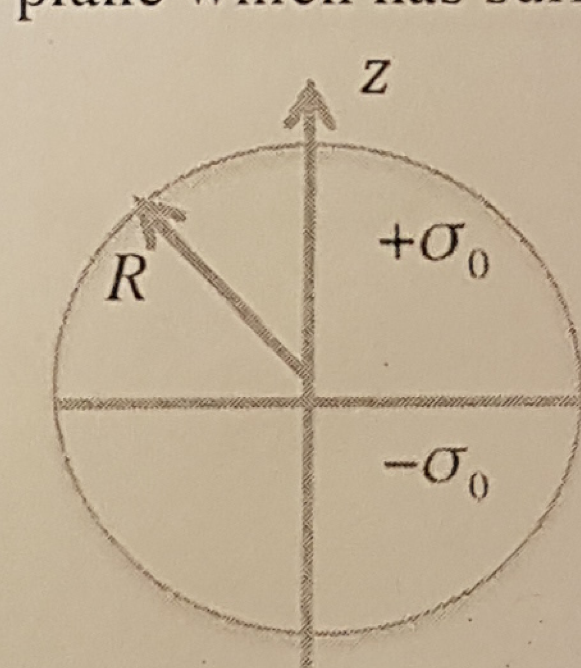
(a) (10 pts) Find the electric field associated with this potential from the gradient of this function in Cartesian coordinates?

(b) (10 pts) What is the divergence of the electric field found in part (a)

(c) (5 pts) The contours of constant f are circles in the (x,y) plane. Draw some arrows indicating the direction of the gradient relative to these contours.

(d) (10 pts) What is the charge density ρ associated with this potential?

20/30 2. Consider a surface charge distribution on a spherical shell of radius R. It is split into two components, one above the z=0 plane which has surface charge density $+\sigma_0$ and one below the $-\sigma_0$, as shown:



10 (a) (10 pts) Find the electric field at the origin.

10 (b) (10 pts) Now find by direct integration the potential associated with this distribution on the z axis for $z>R$.

0 (c) (10 pts) What is the dipole moment associated with this potential?

3. Consider a *spherically* symmetric charge distribution, of the form

$$\rho(r) = a\frac{\rho_0}{r},$$

30

where V and a are constants with the appropriate units.

(a) (10 pts) Using Gauss' law, find the electric field associated with this charge distribution.

(b) (10 pts) What can you say about the electric field as the origin is approached (*i.e.* why is there uncertainty in the direction of field at r=0)?

(c) (10 pts) Setting the potential at the origin to zero, what is the potential in all space? (5 pts) Can you explain why it must diverge as $r\to\infty$?

River Robles

Term One

$$\phi = Ve^{-\left(\frac{x^2+y^2}{2a^2}\right)}$$

(a) $$\vec{E} = -\vec{\nabla}\phi = -\left(\frac{\partial}{\partial x} Ve^{-\frac{x^2+y^2}{2a^2}}\hat{x} + \frac{\partial}{\partial y} Ve^{-\frac{x^2+y^2}{2a^2}}\hat{y}\right)$$

$$= -\left(Ve^{-\frac{x^2+y^2}{2a^2}} \cdot \frac{-2x}{2a^2}\hat{x} + Ve^{-\frac{x^2+y^2}{2a^2}} \cdot \frac{-2y}{2a^2}\hat{y}\right) = \boxed{\frac{V}{a^2} e^{-\left(\frac{x^2+y^2}{2a^2}\right)} (x\hat{x} + y\hat{y})}$$

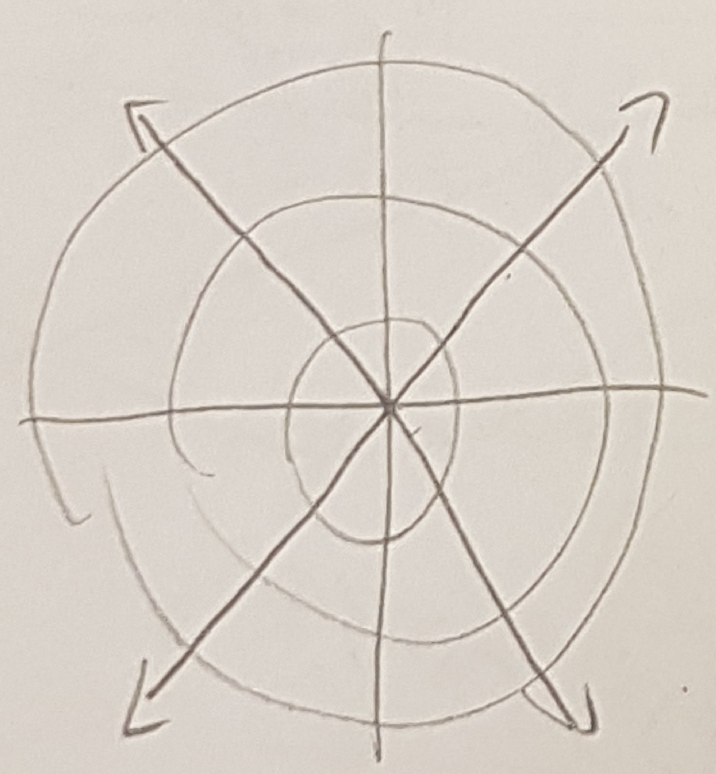
+10

(b) $$\vec{\nabla}\cdot\vec{E} = \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y = \frac{V}{a^2}\left(\frac{\partial}{\partial x} x e^{-\left(\frac{x^2+y^2}{2a^2}\right)} + \frac{\partial}{\partial y} y e^{-\left(\frac{x^2+y^2}{2a^2}\right)}\right)$$

$$= \frac{V}{a^2}\left(e^{-\left(\frac{x^2+y^2}{2a^2}\right)} + x e^{-\left(\frac{x^2+y^2}{2a}\right)}\left(\frac{-2x}{2a^2}\right) + e^{-\left(\frac{x^2+y^2}{2a^2}\right)} + y e^{-\left(\frac{x^2+y^2}{2a^2}\right)} \cdot \left(\frac{-2y}{2a^2}\right)\right)$$

$$= \frac{V}{a^2} e^{-\left(\frac{x^2+y^2}{2a^2}\right)}\left(1 - \frac{x}{a^2} + 1 - \frac{y}{a^2}\right) = \boxed{\frac{V}{a^2} e^{-\left(\frac{x^2+y^2}{2a^2}\right)}\left(2 - \left(\frac{x+y}{a^2}\right)\right)}$$

+10

(c) 

$x\hat{x} + y\hat{y}$ is ⊥ to circles.

+3

(d) $$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \frac{V}{a^2} e^{-\left(\frac{x^2+y^2}{2a^2}\right)}\left(2 - \left(\frac{x+y}{a^2}\right)\right) = \frac{\rho}{\varepsilon_0}$$

$$\rightarrow \boxed{\rho = \frac{V\varepsilon_0}{a^2} e^{-\left(\frac{x^2+y^2}{2a^2}\right)}\left(2 - \left(\frac{x+y}{a^2}\right)\right)}$$

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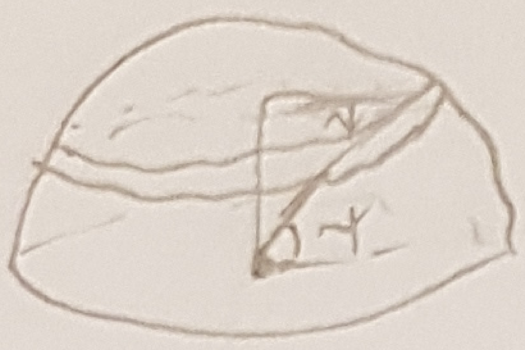
2

(a) 10/10

$$E_{ring} = \frac{Q_{ring}\, z}{4\pi\varepsilon_0 (z^2+r^2)^{3/2}}$$



ring: radius r, thickness $\sqrt{z^2+r^2}\, d\psi$

$\Rightarrow da = 2\pi r \sqrt{z^2+r^2}\, d\psi \Rightarrow Q_{ring} = \sigma \cdot 2\pi r \sqrt{r^2+z^2}\, d\psi$.

$$\Rightarrow dE = \frac{2\pi\sigma r\sqrt{r^2+z^2}\, z\, d\psi}{4\pi\varepsilon_0 (z^2+r^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0}\frac{rz}{r^2+z^2}d\psi$$

$\sin\psi = \frac{z}{\sqrt{r^2+z^2}}$

$\cos\psi = \frac{r}{\sqrt{r^2+z^2}}$

$$= \frac{\sigma}{2\varepsilon_0} \cdot \sin\psi\cos\psi\, d\psi$$

$$\text{so } \vec{E} = \int_0^{\pi/2} \frac{\sigma}{2\varepsilon_0}\sin\psi\cos\psi\, d\psi = \frac{\sigma}{2\varepsilon_0}\cdot\frac{1}{2}\sin^2\psi\Big|_0^{\pi/2}$$

$$= \frac{\sigma}{2\varepsilon_0}\cdot\frac{1}{2}\left(\sin^2\left(\frac{\pi}{2}\right) - \sin^2 0\right) = \frac{\sigma}{4\varepsilon_0}(-\hat{z})$$

Symmetry gives the field from the bottom to be $-\frac{\sigma}{4\varepsilon_0}\hat{z}$

so $\boxed{\vec{E}_{total} = -\frac{\sigma}{2\varepsilon_0}\hat{z}}$ ✓

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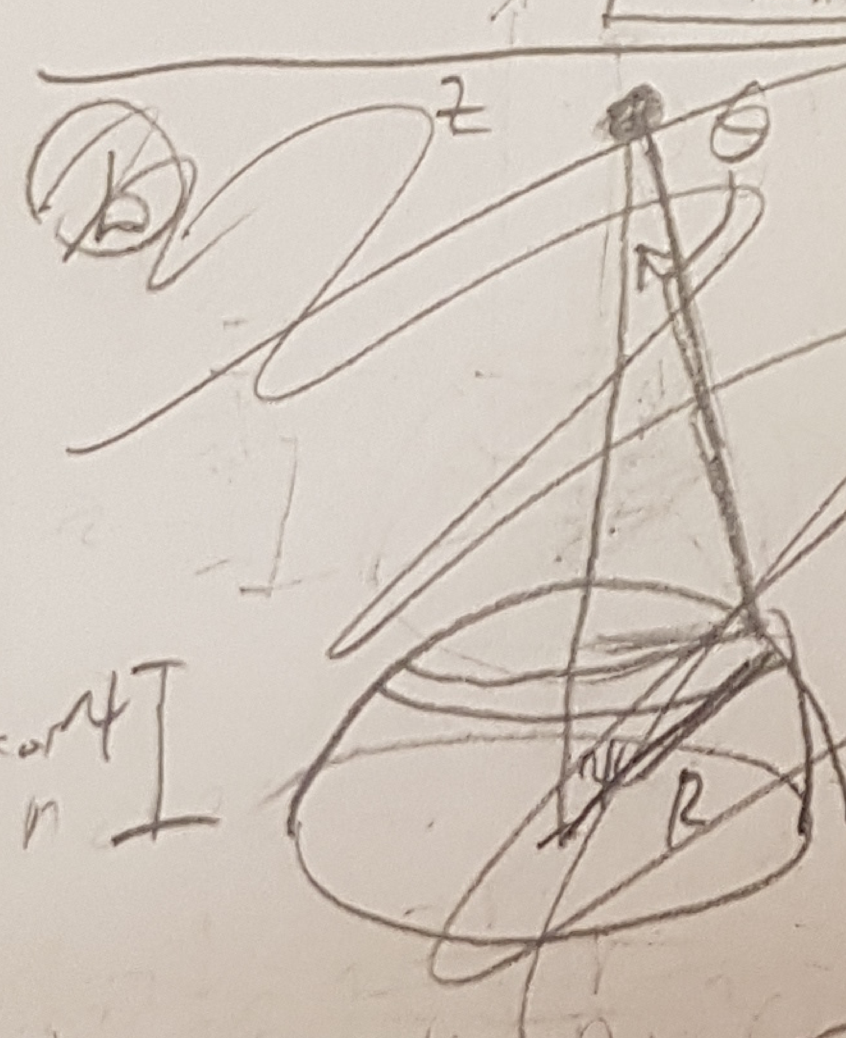
(b)



ring radius $= R\sin\psi$ ring thickness $= R\,d\psi$

so $da = 2\pi R\sin\psi \cdot R\, d\psi$

$$\Rightarrow dE_{ring} = \frac{2\pi R^2\sigma \sin\psi\, (z - R\cos\psi)}{4\pi\varepsilon_0 ((z-R\cos\psi)^2 + (R\sin\psi)^2)^{3/2}} d\psi$$

$$dE = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{(z-R\cos\psi)(\sin\psi)}{((z-R\cos\psi)^2+(R\sin\psi)^2)^{3/2}}\right)d\psi.$$

$(z-R\cos\psi)^2 + (R\sin\psi)^2 = z^2+R^2-2zR\cos\psi$

law of cosines

$$= \frac{\sigma R}{2\varepsilon_0}\left(\frac{(z-R\cos\psi)(R\sin\psi)}{(z^2+R^2-2zR\cos\psi)^{3/2}}\right)d\psi$$

let $u = z^2+R^2-2zR\cos\psi$

$du = +2zR\sin\psi\, d\psi$

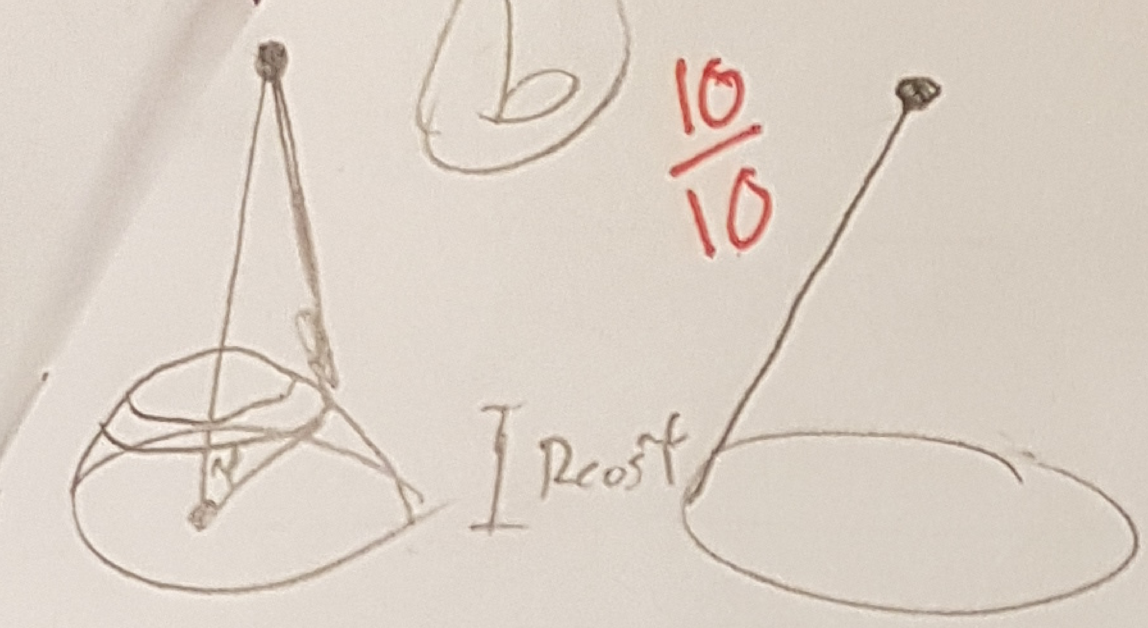
$\frac{u-z^2-R^2}{+2z} = -R\cos\psi$

$$= \frac{\sigma R}{2\varepsilon_0}\left(\frac{z+\frac{u-z^2-R^2}{2z}}{u^{3/2}}\right)\cdot\frac{1}{2z}du$$

$$= \frac{\sigma R}{4\varepsilon_0 z}\left(\frac{2z^2+u-z^2-R^2}{2z\,u^{3/2}}\right)du$$

$\sigma R\left(z^2-R^2+u\right) du$ cont. on other

(b) 10/10



$$\phi_{ring} = \int \frac{dq}{4\pi\varepsilon_0\sqrt{r^2+z^2}} = \frac{Q_{ring}}{4\pi\varepsilon_0\sqrt{r^2+z^2}}$$

ring radius $= R\sin\psi$ ring thickness $= R\,d\psi$

so $da = 2\pi R^2 \sin\psi\, d\psi$

so $$d\phi_{ring} = \frac{2\pi\sigma R^2 \sin\psi\, d\psi}{4\pi\varepsilon_0 \left((z-R\cos\psi)^2+(R\sin\psi)^2\right)^{1/2}}$$

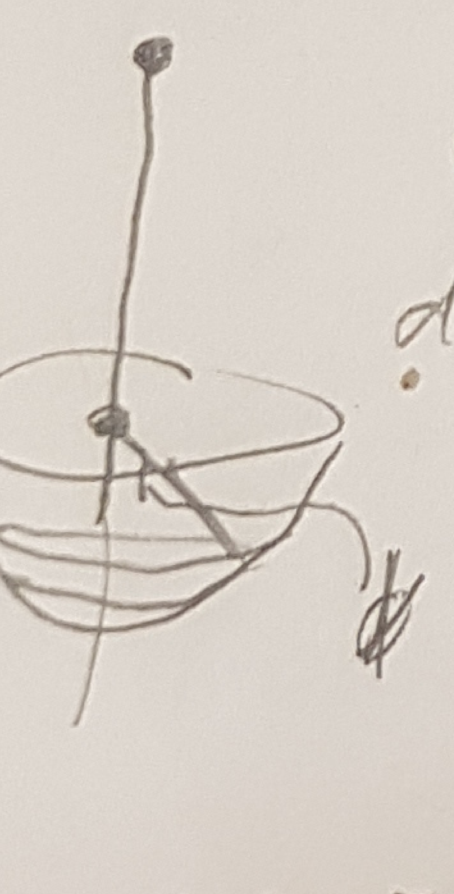
by law of cosines, $(z-R\cos\psi)^2+(R\sin\psi)^2 = z^2+R^2-2zR\cos\psi$

so $$d\phi_{ring} = \frac{\sigma R}{2\varepsilon_0}\frac{R\sin\psi\, d\psi}{(z^2+R^2-2zR\cos\psi)^{1/2}}$$ let $u = R^2+z^2-2zR\cos\psi$

$du = -2zR\sin\psi\, d\psi$

$\Rightarrow d\phi = \frac{\sigma R}{2\varepsilon_0}\cdot\frac{-1}{2z}\frac{du}{u^{1/2}}$ so $\phi = \frac{-\sigma R}{4\varepsilon_0 z}\cdot 2u^{1/2}$

$$\Rightarrow \phi = \frac{-\sigma R}{2\varepsilon_0 z}\sqrt{z^2+R^2-2Rz\cos\psi}\,\Big|_{\frac{\pi}{2}}^{0} = \frac{\sigma R}{2\varepsilon_0 z}\left(\sqrt{z^2+R^2} - \sqrt{z^2+R^2-2Rz}\right)$$



ring radius $= R\sin\phi$ ring thickness $= R\,d\phi$

$da = 2\pi R^2 \sin\phi\, d\phi$

so $$d\phi_{ring} = \frac{-2\pi\sigma R^2\sin\phi\, d\phi}{4\pi\varepsilon_0\sqrt{(z+R\cos\phi)^2+(R\sin\phi)^2}}$$

law of cosines gives

$(z+R\cos\phi)^2+(R\sin\phi)^2 = R^2+z^2-2Rz\cos(\pi-\phi)$

$= R^2+z^2+2Rz\cos\phi$

$$\Rightarrow d\phi = -\frac{\sigma R}{2\varepsilon_0}\frac{R\sin\phi\, d\phi}{(R^2+z^2+2Rz\cos\phi)^{1/2}}$$ let $u = R^2+z^2+2Rz\cos\phi$

$\Rightarrow du = -2Rz\sin\phi\, d\phi$

$$\text{so } d\phi = \frac{-\sigma R}{2\varepsilon_0}\frac{1}{2z}\frac{du}{u^{1/2}} \Rightarrow \phi = \frac{-\sigma R}{4\varepsilon_0 z}2u^{1/2} = \frac{-\sigma R}{2\varepsilon_0 z}\left(\sqrt{R^2+z^2+2Rz\cos\phi}\right)\Big|_{\frac{\pi}{2}}^{0}$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\left(\sqrt{R^2+z^2} - \sqrt{R^2+z^2+2Rz}\right)$$

could simplify

$$\text{so } \boxed{\phi_{total} = \frac{\sigma R}{2\varepsilon_0 z}\left(2\sqrt{R^2+z^2} - \sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)}$$ ✓

(c) reorrange to get

$$\phi_{total} = \frac{2\pi\sigma R z}{4\pi\varepsilon_0 z^2}\left(2\sqrt{z^2+R^2} - \sqrt{z^2+R^2+2Rz} - \sqrt{z^2+R^2-2Rz}\right)$$

which $\Rightarrow$

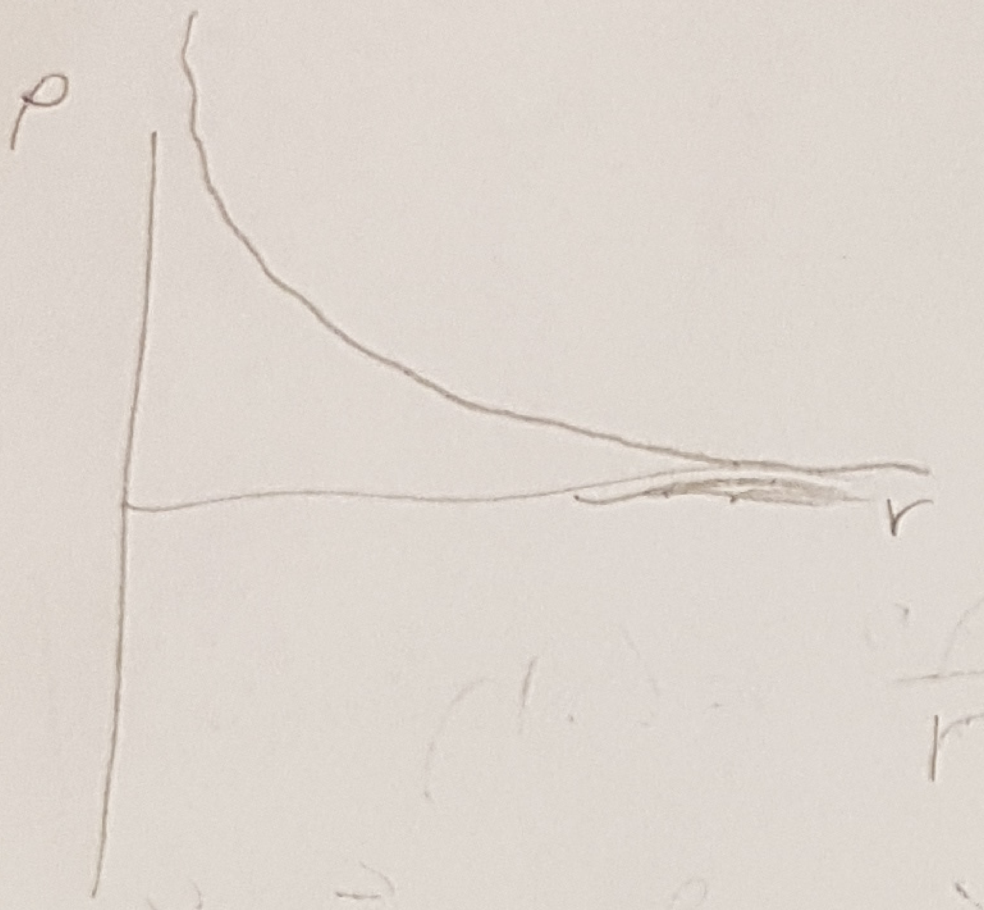
$$p = 2\pi\sigma R z\left(2\sqrt{z^2+R^2} - \sqrt{z^2+R^2+2Rz} - \sqrt{z^2+R^2-2Rz}\right)$$

need to take $z \to \infty$ limit

$\rho(r) = a\frac{\rho_0}{r}$

$\nabla \cdot E = a\frac{\rho_0}{r\varepsilon_0} \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r} r^2 E_r = \frac{a\rho_0}{\varepsilon_0}\frac{1}{r} \Rightarrow \frac{d}{dr} r^2 E_r = \frac{a\rho_0}{\varepsilon_0} r$

$\Rightarrow r^2 E_r = \int \frac{a\rho_0}{\varepsilon_0} r\,dr \Rightarrow r^2 E_r = \frac{a\rho_0}{2\varepsilon_0} r^2 \Rightarrow \boxed{E_r = \frac{a\rho_0}{2\varepsilon_0}\hat{r}}$ HO



b. There is an infinite amount of charge on the origin, so minute changes in x, y, or z lead to dramatic shifts in the direction of the field.

+8

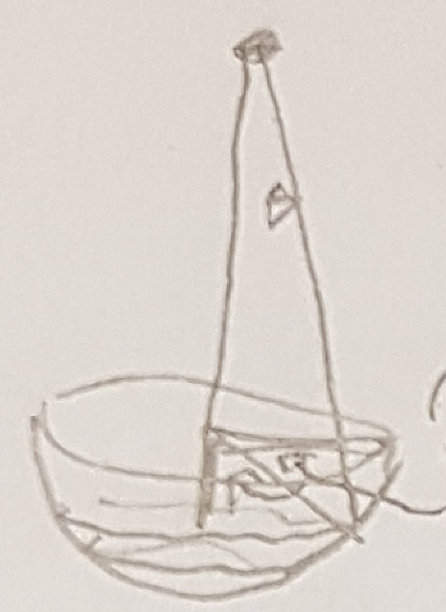
for $r = b$ +5

c. $\phi = -\int_0^b E \cdot dl = -\int_0^b \frac{a\rho_0}{2\varepsilon_0} dr = -\frac{a\rho_0}{2\varepsilon_0}(b-0) = \boxed{\frac{-a\rho_0}{2\varepsilon_0} b}$ +5

The potential must diverge because there is an infinite amount of charge at $r=0$ ($\rho(0) = \frac{\rho_0 a}{0}$) and thus the potential difference between the origin and $r = \infty$ is infinite. Thus making $\phi(0) = 0$ forces $\phi(\infty) = \infty$.

But we defined $\phi(0) = 0$.

+2



radius $= r$ thickness $= \sqrt{r^2+z'^2}\, d\psi$

so $Q_{ring} = \sigma \cdot 2\pi r \sqrt{r^2+z'^2}\, d\psi$

so $dE = \dfrac{\sigma\, 2\pi r \sqrt{r^2+z'^2} \cdot (z+z')\, d\psi}{4\pi\varepsilon_0 \left((z+z')^2 + r^2\right)^{3/2}}$

$\sin\psi = \dfrac{r}{\sqrt{r^2+z'^2}}$

$\cos\psi = \dfrac{z'}{\sqrt{r^2+z'^2}}$ $dE =$