PHYSICS 1B

FINAL EXAM

Spring, 2017 Dr. Coroniti

There are 225 points on the exam, and you have three hours. To receive full credit, show all your work and reasoning. No credit will be given for answers that simply "appear". The exam is closed notes and closed book. You do not need calculators, so please put them, and any cell phones, away. If you need more space, use the backside of the page.

Simon Rufer

Your Full Name - Printed

Simon Rufer

Your Normal Signature

trav $y(x,t) = A\cos(kx-\omega t)$

SHO $A\cos(\omega t+\phi)$

stand $[A\sin kx + B\cos kx]\cos(\omega t+\phi)$

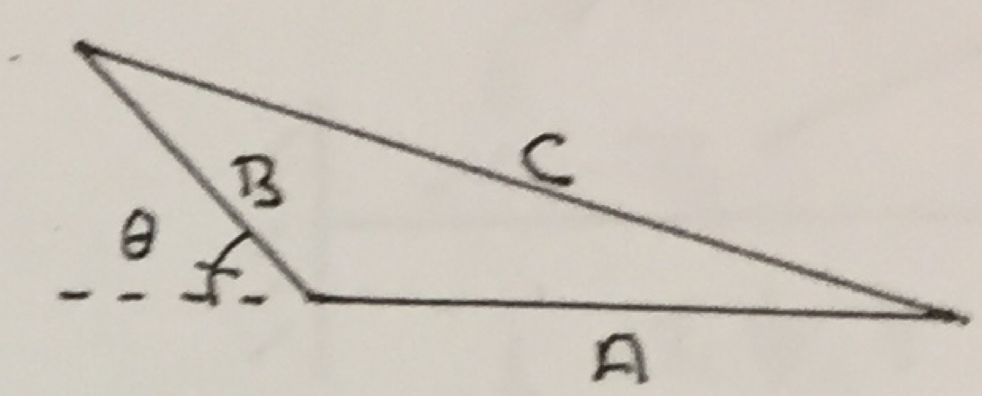
Problem	Score
1	25
2	27
3	21
4	26
5	21
6	13
7	16
8	25
Total	174

FORMULAE

1. $\frac{1}{(1+x)^{1/2}} \approx 1 - \frac{x}{2}$; $x \ll 1$ 2. $\frac{1}{(1 \pm x)^2} \approx 1 \mp 2x$ 3. $(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$

4. $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{\frac{1}{2}}}$ 5. $\int sin^2(\theta) = \frac{\theta}{2} - \frac{\sin(2\theta)}{4}$ $\Big|_0^\pi$ $\frac{\pi}{2} - 0$

6. $< sin^2(\theta) > = < cos^2(\theta) = 1/2$



7. Law of Cosines

$$C = [A^2 + B^2 + 2AB cos(\theta)]^{1/2}$$

8. $\underline{E} = \frac{Q\,\hat{r}}{4\pi\epsilon_0 r^2}$ 9. $d\underline{E} = \frac{dQ\,\hat{r}}{4\pi\epsilon_0 r^2}$ 10. $\oint\!\!\oint \underline{E} \cdot d\underline{A} = \frac{Q}{\epsilon_0}$ 11. $dV = -\underline{E} \cdot \underline{dr}$

12. $dV = \frac{dQ}{4\pi\epsilon_0 r}$ 13. Capacitance $C = \frac{dQ}{dV}$

Parallel $C_{equivalent} = \Sigma_i C_i$ Series $\frac{1}{C_{equivalent}} = \Sigma_i \frac{1}{C_i}$

14. Dielectrics $\kappa\epsilon_0 \iint \underline{\mathbf{E}} \cdot \underline{\mathbf{dA}} = q_{free}$

15. Ohm's Law $V_R = I\,R$

16. Kirchhoff's Laws

Junction Theorem $\Sigma_i I_i = 0$ Loop Theorem $\Sigma_i \Delta V_i = 0$

17. Force on a Current Element $\underline{\mathbf{dF}} = I\underline{\mathbf{dl}} \times \underline{\mathbf{B}}$

18. Magnetic Dipole Moment $\underline{\mu} = (area)\,\hat{\mathbf{n}}$

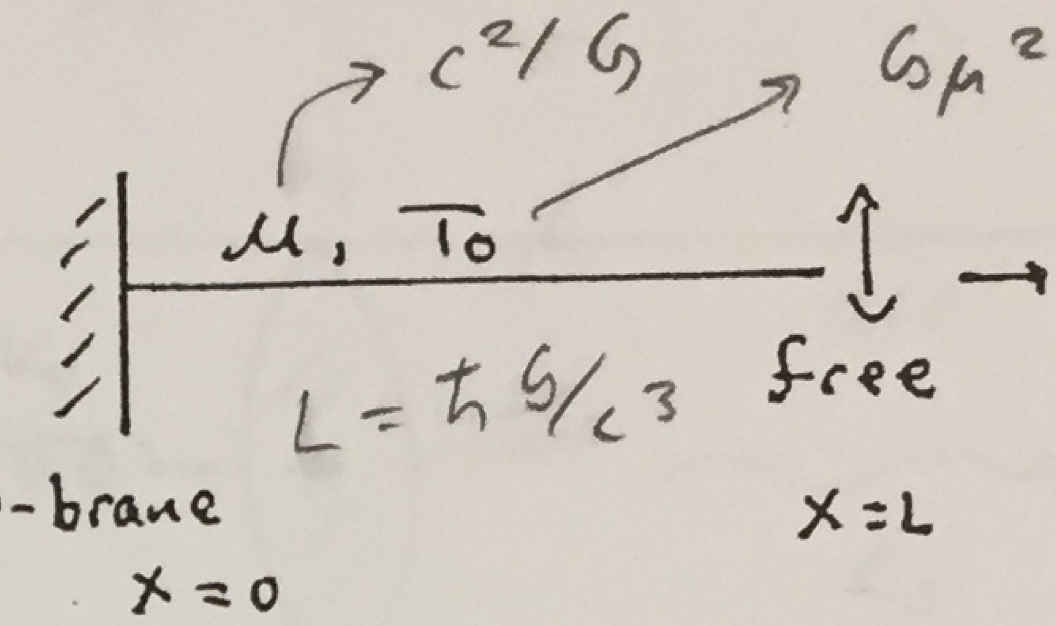
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial y^2}{\partial x^2}$$

$L = \hbar G/c^3 \quad \hbar = h/2\pi$

(25 Pts)

1. In String Theory, the elementary particles (such as electrons and quarks) are described as wave-like oscillations of finite length L pieces of string. (L = Planck length ($\hbar G/c^3$) where $\hbar$ is Planck's constant $h/2\pi$, G is Newton's constant of Universal Gravitation, and c is the speed of light.) These strings have a mass per unit length $\mu = c^2/G$, and an equilibrium tension $T_0 = G\mu^2$. The strings support transverse displacements y (x, t) that obey the classical wave equation with $T_0/\mu = c^2$. Strings can be attached to membranes (called D-branes) so that one end x = 0 is fixed, and the other end x = L is free to oscillate in the x-y plane.

(5) a. Explain why the appropriate boundary conditions for y (x, t) are y (0, t) = 0 and $\partial y/\partial x\,|_{x=L} = 0$.

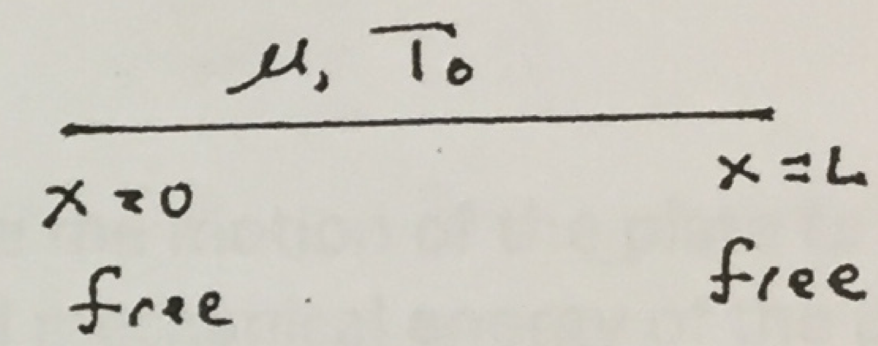


(10) b. By considering sinusoidal oscillations of the string, *prove* that the frequencies of the normal modes are given by

$$\nu_n = \frac{(2n-1)\,c}{4L} \qquad n = 1, 2, 3, \ldots$$

25

(10) c. If the string is detached from the membrane, both ends are free to oscillate. For sinusoidal oscillations, *prove* that the frequencies of the normal modes are $\nu_n = nc/2L$, and sketch the first two normal modes as a function of x.



a) At x=0, the string is fixed to the membrane. Therefore, at all times t, $\boxed{y(0,t) \text{ must equal } 0.}$ At x=L, there is nothing for the string to exert a force upon. Therefore, $\partial y/\partial x$ (which is directly related to the force by $F = -T_0\, \partial y/\partial x$) must be zero: $\boxed{\partial y/\partial x\,|_{x=L} = 0}$

5

b) $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial y^2}{\partial x^2} \Rightarrow$ Solutions: $y(x,t) = [A\sin kx + B\cos kx]\cos\omega t$

10

$y(0,t) = 0$: x trig term must be $\sin kx$, so $y(x,t) = A\sin kx \cos\omega t$

$\frac{\partial y}{\partial x}\Big|_{x=L} = 0$: $Ak\cos kx \cos\omega t = 0$ for $x = L$

$\cos kx = 0 \Rightarrow kx = \frac{(2n-1)\pi}{2}$ $\quad k = \frac{(2n-1)\pi}{2L}$ $\quad \omega = kc$ $\quad \omega = \frac{(2n-1)\pi c}{2L}$

$\omega = \frac{2\pi}{T}$ $\quad T = \frac{1}{f}$ $\quad \omega = 2\pi\nu$ $\quad \nu = \frac{\omega}{2\pi}$

$$\boxed{\nu = \frac{(2n-1)}{4L} c}$$

c) $\frac{\partial y}{\partial x}\Big|_{x=0,\,x=L} = 0$

From boundary conditions: $[A\sin kx + B\cos kx]\cos\omega t$

$y(x,t) = B\cos kx \cos\omega t$

$\frac{\partial y}{\partial x} = -Bk\sin kx \cos\omega t$

for $\frac{\partial y}{\partial x}\Big|_{x=0,\,x=L} = 0$

$kx = n\pi$

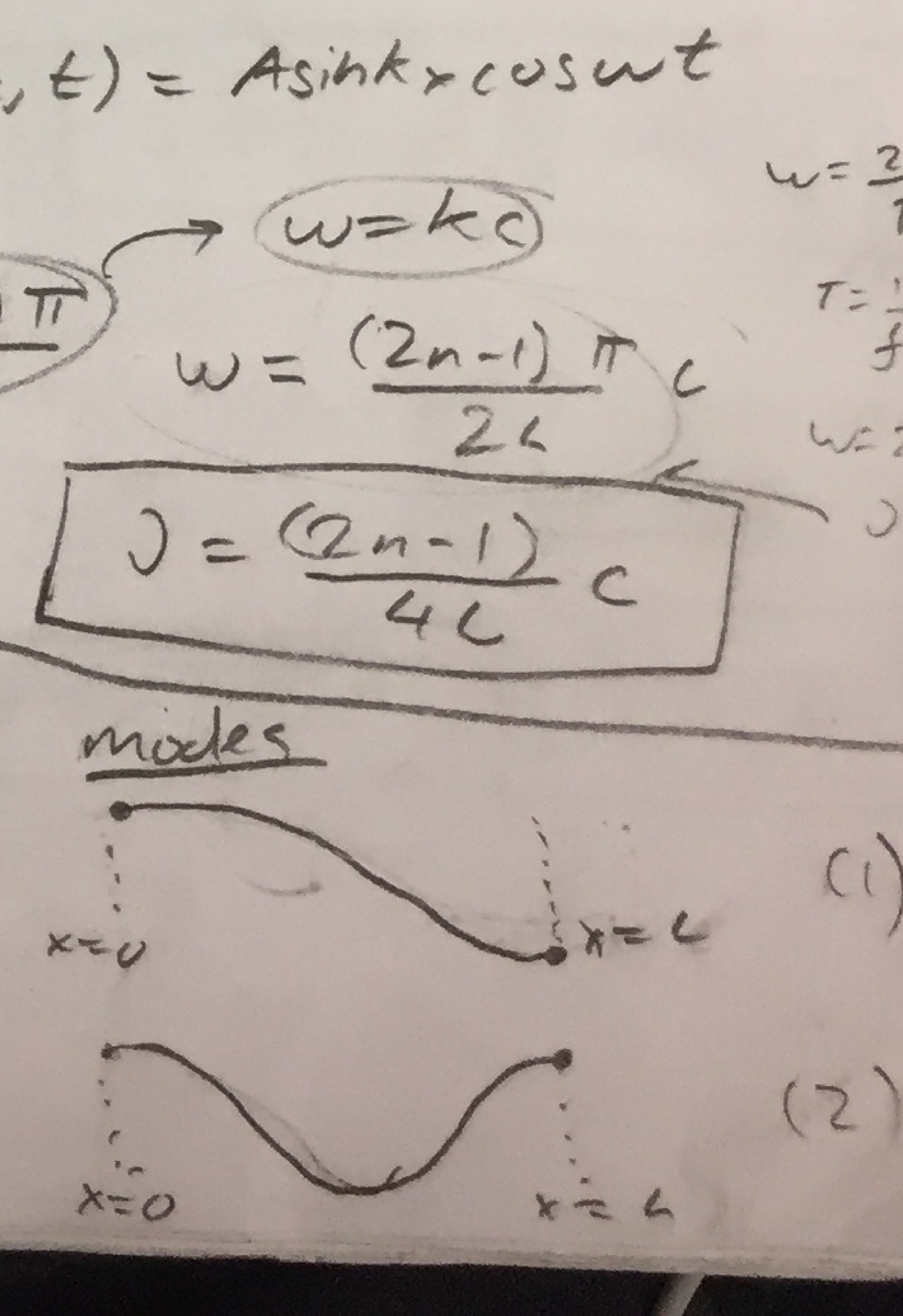
$k = \frac{n\pi}{L}$

$\omega = kc$

$\omega = \frac{n\pi}{L} c$ $\quad \nu = \frac{\omega}{2\pi}$

$$\boxed{\nu = \frac{nc}{2L}}$$

modes



$t=0$

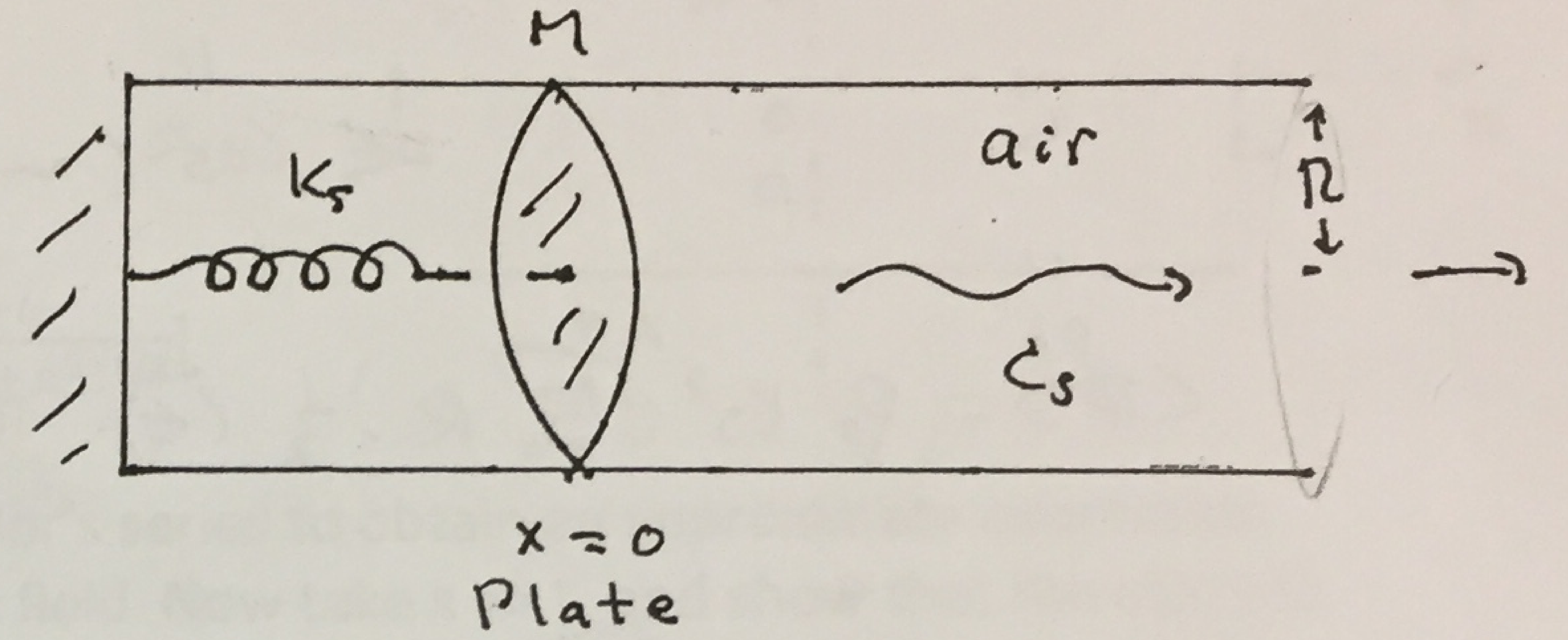
$x=0$

(30 Pts)

2. A spring with stiffness constant k_S has one end attached to a wall and the other end attached to a circular plate with ~~radius R and mass M~~. The plate is free to slide without friction inside a long cylindrical pipe with the same radius. The pipe contains air having a mass density ρ_0 and speed of sound C_s . The plate oscillates about its equilibrium position x = 0 with an amplitude A , and, at time t = 0, the plate is observed to be at its maximum positive extension ~~away from x = 0~~. Express your answers in terms of the given variables k_S , M, ρ_0 , C_s , A, and R.

(10) a. What is the angular frequency of the plate's oscillation? What are the position, velocity of the plate as functions of time for t > 0?

k_s, R, M



(10) b. The oscillations of the plate (a loud speaker) generate sound waves that travel down the pipe. What are the angular frequency and wavenumber of the sound waves? Find expressions for the longitudinal displacement $y\,(x,t)$ and the perturbed pressure $\delta P\,(x,t)$ of the air due to the sound waves? [Recall: $\delta P = -\rho_0 C_S^2\ \partial y/\partial x$.]

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(10) c. The sound wave carry away energy which will cause the motion of the plate to damp. (This is called radiation damping.) Consider the total mechanical energy of the plate E. Write an expression for the time-averaged power (= Intensity x area, $I = \delta P\ \partial y/\partial t$) carried away by the sound waves expressed in terms of the amplitude of the air's displacement y (x, t). Let the amplitude of the plate A be a function of time (i.e., A = A (t)) due to the radiation damping. Differentiate E with respect to time, and write an equation which balances the loss of total mechanical energy with the power carried by the sound wave. Show that the characteristic damping time τ of the plate is given by

$a=\omega^2 x$

$$\tau = \frac{M}{\pi R^2\, \rho_0\, C_S}.$$ The damping is exponential so that $A^2(t) = A^2(0)\exp[-t/\tau]$.

a) 10 $\omega = \sqrt{\frac{k}{m}}$ $\boxed{\omega = \sqrt{\frac{k_s}{M}}}$ General form: $x(t) = A\cos(\omega t + \phi)$ $A = x$ $\sqrt{\frac{k_s}{m}}$

$\boxed{x(t) = A\cos\left(\sqrt{\frac{k_s}{M}}\,t\right) \quad \frac{\partial y}{\partial x} = v(t) = -A\sqrt{\frac{k_s}{M}}\sin\left(\sqrt{\frac{k_s}{M}}\,t\right)}$

b) $\omega = \sqrt{\frac{k_s}{M}}$ $\delta P = -\rho_0 C_s^2 \frac{\partial y}{\partial x}$

traveling sound waves: $y(x,t) = A\cos(kx - \omega t)$ $\partial y/\partial x = -Ak\sin(kx - \omega t) = \sin kx \sin \omega t$

$\omega = kc$, $k = \omega/c = \sqrt{\frac{k_s}{m}}/c$ $\boxed{\omega = \sqrt{\frac{k_s}{m}}}$ $\boxed{k = \sqrt{\frac{k_s}{m}}/c_s}$

amplitude of wave $\boxed{y(x,t) = R\cos\left(\sqrt{\frac{k_s}{m}}/C_s\, x - \sqrt{\frac{k_s}{m}}\,t\right)}$ $\boxed{\delta P(x,t) = +\rho_0 C_s R\sqrt{\frac{k_s}{m}}\sin\left(\sqrt{\frac{k_s}{m}}/c_s\, x - \sqrt{\frac{k_s}{m}}\,t\right)}$

$\boxed{\text{c) on back}}$

c) $E = \frac{1}{2} k_s A^2$

$E = \frac{1}{2} k_s A(t)^2$

$dE = \frac{1}{\cancel{2}} k_s A(t)$

$\langle P \rangle = (\text{Intensity} \times \text{Area})$ $\quad I = \delta P \, \delta y / \delta t$

$\langle P \rangle = \left(\rho_0 c_s^2 \sqrt{\frac{k_s}{m}} R \cos\left(\sqrt{\frac{k_s}{m}} / c_s x - \sqrt{\frac{k_s}{m}} t \right) \right) \pi R^2$

$$\begin{array}{c} \| \\ \langle \cos^2(\sim) \rangle = \frac{1}{2} \end{array}$$

$\langle P \rangle = \left(\rho_0 c_s^2 \sqrt{\frac{k_s}{m}} R \right) \frac{1}{2} \frac{1}{(t)} \pi r^2$

$dE = \langle P \rangle$

$k_s A(t) = \left(\rho_0 c_s^2 \sqrt{\frac{k_s}{m}} R \right) \frac{1}{2} \frac{1}{(t)} \pi r^2$

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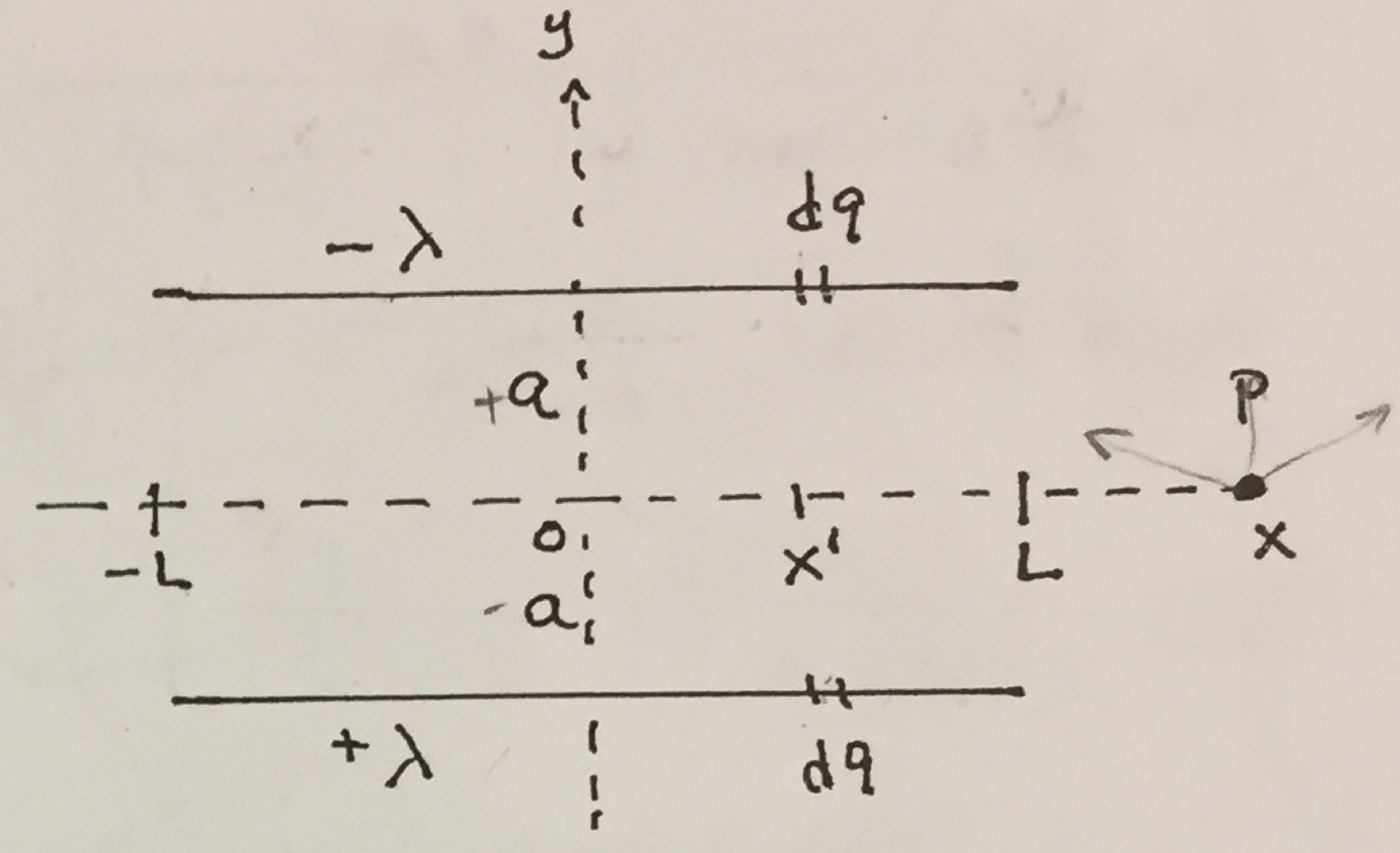
(28 Pts)

3. A thin rod with a charge per unit length $-\lambda$ runs parallel to the x'–axis at y = + a, and extends from x' = - L to x' = + L as shown. An identical rod with a charge per unit length $+\lambda$ runs parallel to the x'-axis at y = - a, and also extends from x' = - L to x' = + L as shown. The electric field that is produced by this charge distribution is to be calculated at a point P which is at a distance x from the origin along the x'-axis.



(8) a. For each rod, identify a differential charge dq located at an arbitrary distance x'. Now find the direction and magnitude of the total differential electric field at P produced by dq.

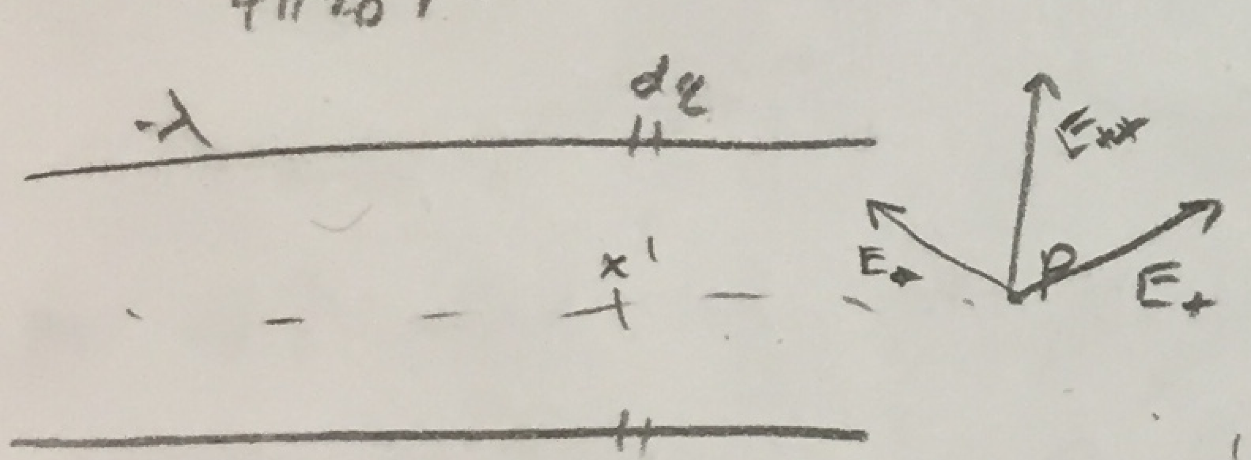
(10) b. Show that the magnitude of the total electric field at P is

$$E = \frac{\lambda}{2\pi\epsilon_0 a}\left[\frac{x+L}{((x+L)^2+a^2)^{1/2}} - \frac{x-L}{((x-L)^2+a^2)^{1/2}}\right]$$

(10) c. First consider $x \pm L \gg a$, and use Taylor's series to obtain an approximate expression for the magnitude for the total electric field. Now take x >> L, and show that the electric field is approximately given by

$$E \approx \frac{L\lambda a}{\pi\epsilon_0 x^3}$$ Explain why this result is reasonable.

a) $dE = \frac{dq\,\hat{r}}{4\pi\epsilon_0 r^2}$ $dq = \lambda\, dx$



$\boxed{dE_x = 0, \text{ by symmetry}}$

$dE_y = dE_{y-} + dE_{y+} = 2(dE_{y-})\sin\theta$

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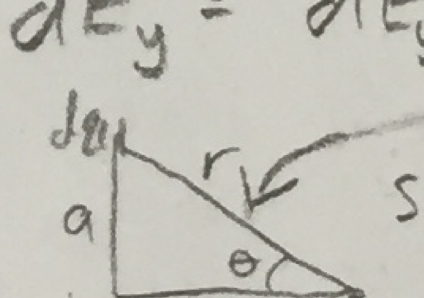


$\sin\theta = \frac{a}{r}$

$$\boxed{dE_{dq} = 2\left(\frac{\lambda\, dx'}{4\pi\epsilon_0 r^2}\right)\frac{a}{r}\hat{y} = \frac{2a\lambda\, dx'}{4\pi\epsilon_0 (a^2+(x-x')^2)^{3/2}}\ (\hat{y})}$$

b) $E_{tot} = \int_{x'=-L}^{x'=L} dE_{dq}$

10

$$E_{tot} = \int_{x'=-L}^{x'=L} \frac{2a\lambda\, dx'}{4\pi\epsilon_0 (a^2+(x-x')^2)^{3/2}} = \frac{2a\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dx'}{(a^2+(x-x')^2)^{3/2}} = \frac{2a\lambda}{4\pi\epsilon_0}\left[\frac{x-x'}{a^2((x-x')^2+a^2)^{1/2}}\right]_{x'=-L}^{x'=L}$$

$$E_{tot} = \frac{\cancel{2}\cancel{a}\lambda}{\cancel{2}4\pi\epsilon_0 a^{\cancel{2}}}\left(\frac{x+L}{((x+L)^2+a^2)^{1/2}} - \frac{x-L}{((x-L)^2+a^2)^{1/2}}\right)$$

$\boxed{\text{c) on back}}$

$$\boxed{E_{tot} = \frac{\lambda}{2\pi\epsilon_0 a}\left[\frac{x+L}{((x+L)^2+a^2)^{1/2}} - \frac{x-L}{((x-L)^2+a^2)^{1/2}}\right]}$$

c) $E_{tot} = \frac{\lambda}{2\pi\varepsilon_0 a}\left[\frac{x+L}{((x+L)^2+a^2)^{1/2}} - \frac{x-L}{((x-L)^2+a^2)^{1/2}}\right]$

$\frac{\lambda}{2\pi\varepsilon_0 a(x-L)(x+L)}\left[\frac{x+L}{\left(\frac{1}{(x-L)^2}+\frac{a^2}{(x+L)(x-L)}\right.}\right.$

use taylor series for $x \pm L >> a$
↳ want this on bottom

$E_{x+L} = \frac{\lambda}{2\pi\varepsilon_0 a\cancel{(x+L)}}\left[\frac{\cancel{x+L}}{\left(1+\left(\frac{a}{x+L}\right)^2\right)^{1/2}}\right]$

taylor series: $\frac{1}{(1+x)^{1/2}} = 1 - \frac{x}{2}$

$E_{x+L} = \frac{\lambda}{2\pi\varepsilon_0 a}\left[1 - \cancelto{0}{\frac{a^2}{2(x+L)^2}}\right]$

3

$E_{x+L} = \frac{L\lambda}{2\pi\varepsilon_0 a^2} \approx \frac{L\lambda a}{2\pi\varepsilon_0 x^3}$

This result makes sense, because if the point is very far away from the rods, the electric field is as if there are two point charges of charge $L\lambda$. However, the electric field is still small because of the factor of a on top.

$= \frac{\pi a^2 \rho L}{\varepsilon_0}$ $\frac{a\rho}{2\varepsilon_0}$

total charge $= \lambda L = \pi a^2 \rho$

diff charge $= \pi r^2 \rho L$

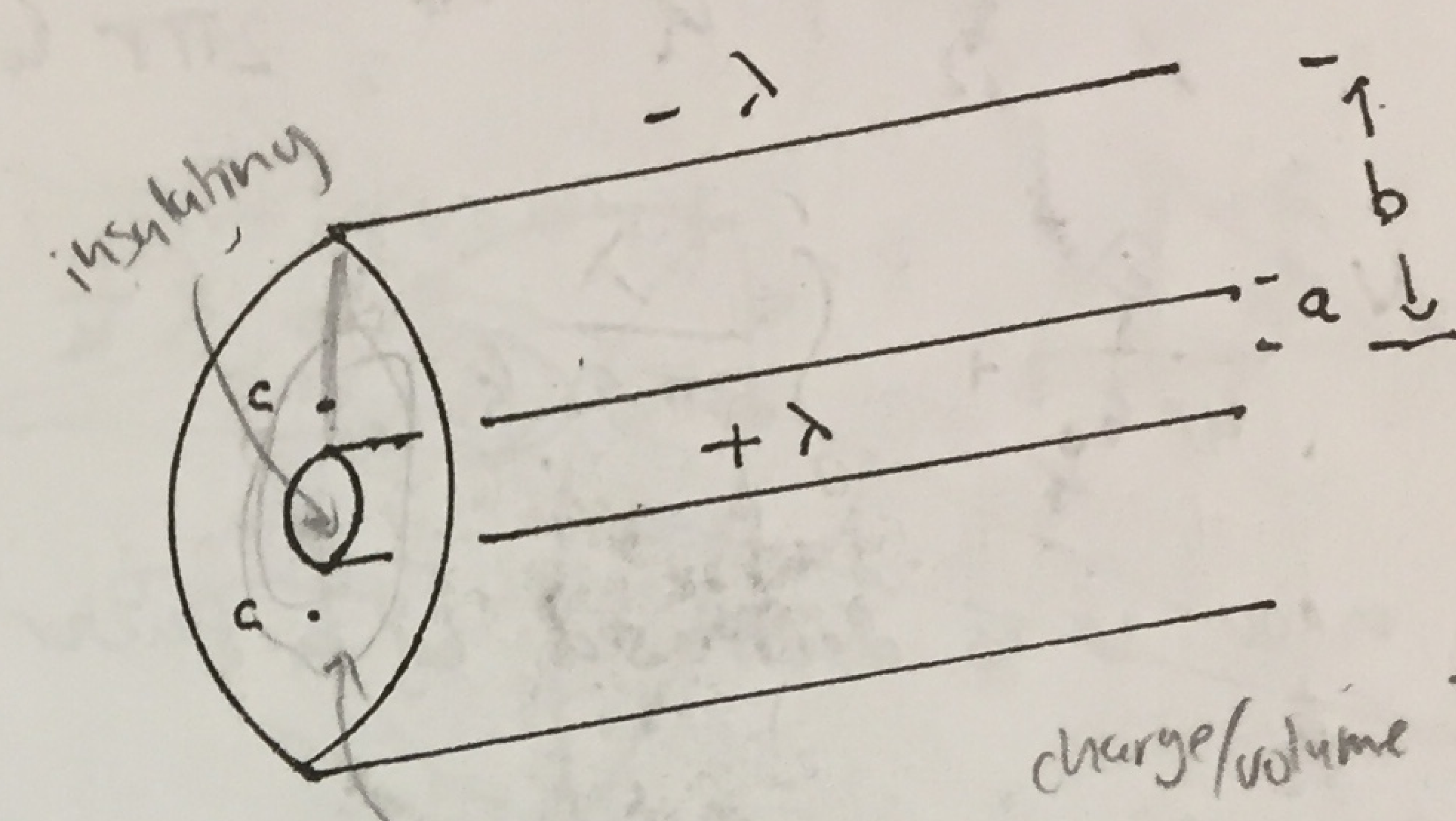
(30 Pts)

4. A very long, insulating rod with a circular radius a has a positive charge per unit volume ρ so as to produce a uniform charge per unit length $\lambda = \pi a^2 \rho$. The rod is surrounded by a thin, cylindrical conducting shell with radius b (> a) that carries a uniform negative charge per unit length $-\lambda$.

(10) a. Use Gauss's Law to find the electric field in the regions:
   (i) 0 < r < a
   (ii) a < r < b
   (III) b < r

(10) b. Taking the zero of the electric potential V (r) to be at r = 0, find V(r) for the regions:
   (i) 0 < r < a
   (ii) a < r < b

(10) c. The region $a \le r \le c < b$ is now filled with a dielectric material with dielectric constant κ. Show that the capacitance per unit length (dC/dl) is given by



charge/volume $= \rho$

$\rho \cdot$ volume!

$(\pi r^2) L \cdot \rho =$ charge

$$\frac{dC}{dl} = \frac{2\pi\epsilon_0}{[\frac{1}{2} + \frac{1}{\kappa}\ln(\frac{c}{a}) + \ln(\frac{b}{c})]}$$

$\rho = \frac{\lambda}{\pi a^2}$

a) i) $0 < r < a$ $\oint E \cdot dA = \frac{q_{in}}{\varepsilon_0} = \frac{\pi r^2 L \rho}{\varepsilon_0}$ $q_{in} = \pi r^2 L \rho L$

$E \cdot 2\pi r L = \frac{\pi r^2 L \rho}{\varepsilon_0}$ $\boxed{E_{(i)} = \frac{r\rho}{2\varepsilon_0} = \frac{r\lambda}{2\pi a^2 \varepsilon_0}}$ ✓

$\rho = \frac{Q}{\pi a^2}$

ii) $a < r < b$

$\oint E \cdot dA = \frac{q_{in}}{\varepsilon_0}$ $E \cdot 2\pi r L = \frac{\lambda L}{\varepsilon_0}$ $\boxed{E = \frac{\lambda}{2\pi r \varepsilon_0}}$ ✓

iii) $r > b$ $q_{in} = 0$ $\therefore \boxed{E = 0}$ ✓

b) i) $0 < r < a$

$V = -\int E\,dr = \int_0^r \frac{r\lambda}{2\pi a^2 \varepsilon_0} dr = \frac{1}{2}\frac{\lambda}{2\pi a^2 \varepsilon_0}(r^2)\Big|_0^r$ $\boxed{V_{(i)} = \frac{\lambda r^2}{4\pi a^2 \varepsilon_0}}$ 9

ii) $0 < r < b$

$V = \int E\,dr = \int_0^a \frac{r\lambda}{2\pi a^2 \varepsilon_0} + \int_a^r \frac{\lambda}{2\pi r \varepsilon_0} = \frac{\lambda}{4\pi\varepsilon_0} + \frac{\lambda}{2\pi\varepsilon_0}(\ln(r)\Big|_a^r)$

$\boxed{V(r) = \frac{\lambda}{4\pi\varepsilon_0} + \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r}{a}\right)}$

c) on back

(30 Pts)

c) Show $dC/d\ell = \dfrac{2\pi\varepsilon_0}{\left[\frac{1}{2} + \frac{1}{k}\ln\left(\frac{c}{a}\right) + \ln\left(\frac{b}{c}\right)\right]}$

$$E_{field}\ \text{inside} = \frac{E_{before}}{k} = \frac{\lambda}{2\pi r \varepsilon_0 k}$$

new V: $\dfrac{\lambda}{4\pi\varepsilon_0} + \displaystyle\int_0^r \frac{\lambda}{2\pi\varepsilon_0 k} = \frac{1}{k}\ln\left(\frac{c}{a}\right) + \frac{1}{k}\ln\left(\frac{b}{c}\right)$

new V is decreased by a factor of $\frac{1}{k}\ln\left(\frac{c}{a}\right) + \ln\left(\frac{b}{c}\right)$

$Q = CV$

$$V = \frac{Q}{C}$$

original capacitance: $Q = CV$

$$V = \frac{\lambda}{4\pi\varepsilon_0} + \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r}{a}\right)$$

Q

$$V = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2} + \ln\left(\frac{r}{a}\right)\right)$$

$$\frac{dC}{d\ell} = \frac{2\pi\varepsilon_0}{2 + \ln\left(\frac{r}{a}\right)}$$ original capacitance

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$$V_{new} = \frac{\lambda}{4\pi\varepsilon_0 k} + \frac{\lambda}{2\pi\varepsilon_0 k}\int_c^a \frac{1}{r} + \int_b^c \frac{1}{r}$$

$$V_{new} = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{2} + \frac{1}{k}\ln\left(\frac{c}{a}\right) + \frac{1}{k}\ln\left(\frac{b}{c}\right)\right)$$

$$\boxed{\frac{dC_{new}}{d\ell} = \frac{2\pi\varepsilon_0}{\left(\frac{1}{2} + \frac{1}{k}\ln\left(\frac{c}{a}\right) + \frac{1}{k}\ln\left(\frac{b}{c}\right)\right)}}$$

(30 Pts)

5. A thin hemispherical shell of radius a carries a charge +Q that is uniformly distributed over its surface (surface charge density $\sigma = Q/2\pi a^2$). We wish to calculate the electric potential at a Point P that is a distance z from the center of the shell along the axis of symmetry as shown.

(10) a. Identify a differential element of charge dq, and find the contribution of dq to potential. [Recall: the area element on the surface of a sphere is $dA = r^2 \sin(\theta)\, d\theta\, d\phi$, and the Law of Cosines.]

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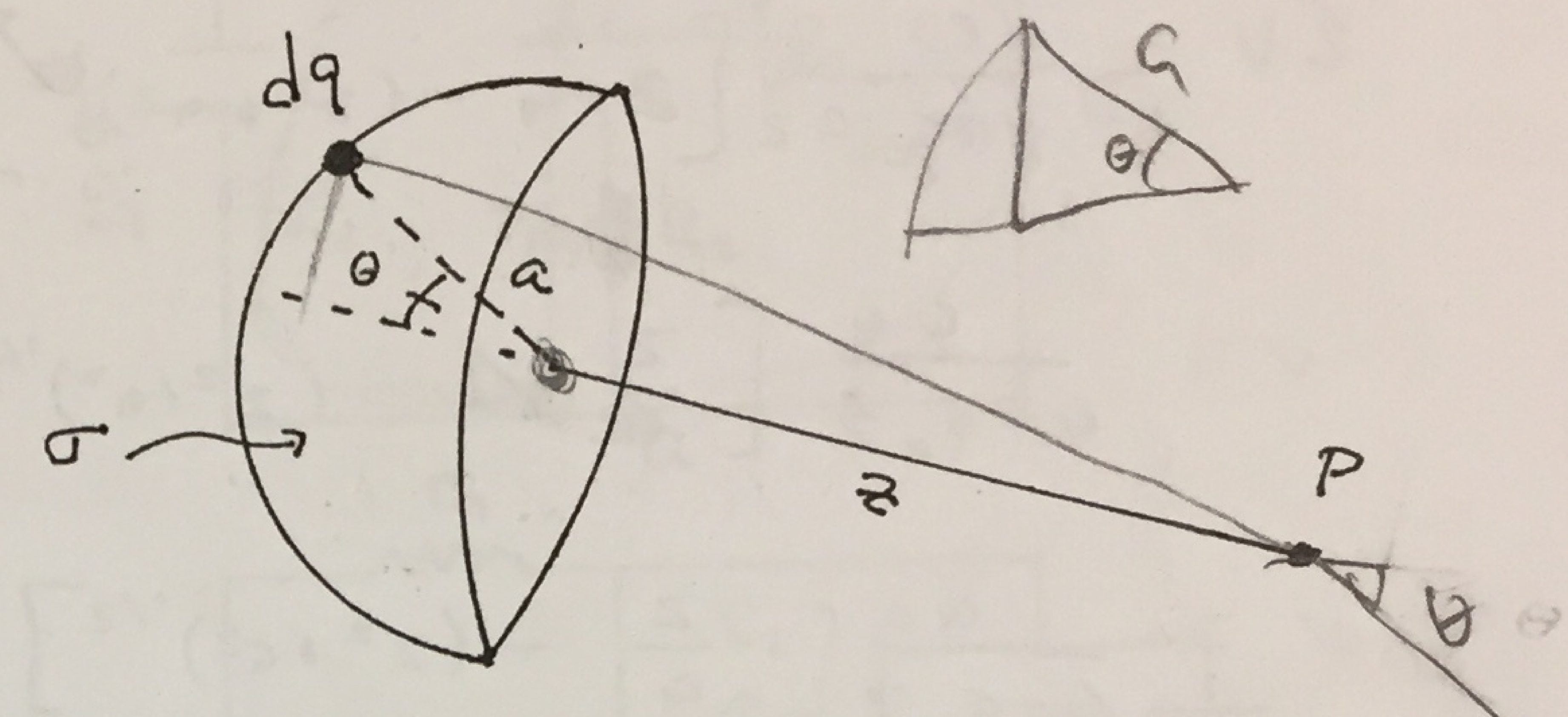

(10) b. Show that the total electric potential at P is given by

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$$V = \frac{Q}{4\pi\epsilon_0 z a}\left[z + a - (z^2 + a^2)^{\frac{1}{2}}\right]$$

cosθ

(10) c. Now suppose that a point charge - Q is placed at the origin. If z >> a, show that the total electric potential that is produced by the resulting charge distribution is approximately given by

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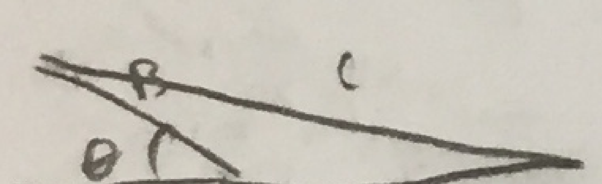
$$V \approx -\frac{Qa}{8\pi\epsilon_0 z^2}$$

What type of electric potential does V represent?

a)



$\left[A^2 + B^2 + 2AB\cos\theta\right]^{1/2} = C = r$

$r = \left[a^2 + z^2 + 2az\cos\theta\right]^{1/2}$

$dq = \sigma dA$

$$\boxed{dV = \frac{dq}{4\pi\epsilon_0 r} = \frac{\sigma\, dA \cos\theta}{4\pi\epsilon_0\left[a^2+z^2+2az\cos\theta\right]^{1/2}} = \frac{\sigma a^2 \sin\theta \cos\theta\, d\theta\, d\phi}{4\pi\epsilon_0\left[a^2+z^2+2az\cos\theta\right]^{1/2}}}$$

b) $V = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi} \frac{\overset{\to Q/2\pi a^2}{\sigma}\, a^2 \sin\theta\, d\theta\, d\phi\, \cos\theta}{4\pi\epsilon_0\left[a^2+z^2+2az\cos\theta\right]^{1/2}}$

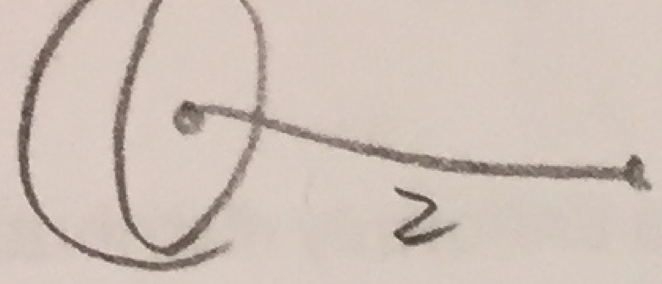
$V = \pi \int_0^{2\pi} \frac{Q}{8\pi^2\epsilon_0}\, \frac{\sin\theta\cos\theta\, d\theta}{\left[a^2+z^2+2az\cos\theta\right]^{1/2}}$

$V = \frac{Q}{8\pi\epsilon_0}\int_0^{2\pi} \frac{\overbrace{\sin\theta\cos\theta}^{\to \frac{\sin 2\theta}{2}}\, d\theta}{\left[a^2+z^2+2az\cos\theta\right]^{1/2}} = \frac{Q}{8\pi\epsilon_0 az}\left[z + a - (z^2+a^2)^{1/2}\right]\int_0^{2\pi} \cancel{\frac{\sin 2\theta}{2}} \to 2$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 az}\left[z + a - (z^2+a^2)^{1/2}\right]}$$

c) on back!

c)



$$V(\text{pt-charge}) = \frac{-Q}{4\pi\varepsilon_0 z}$$

$$\Sigma V = \frac{Q}{4\pi\varepsilon_0 a z}\left[z + a - (z^2 + a^2)^{1/2}\right] - \frac{Qa}{4\pi\varepsilon_0 a z}$$

$$V = \frac{Qa}{4\pi\varepsilon_0 z}\left[\frac{z}{a} + \cancel{1} - (z^2+a^2)^{1/2} - \cancel{1}\right]$$

$$V = \frac{Qa}{4\pi\varepsilon_0 z}\left[\frac{z}{a} - (z^2+a^2)^{1/2}\right] \quad z >> a$$

↳ want z in denom

→ factor out z!

$$V = \frac{Qa}{4\pi\varepsilon_0 z^2}\left[\frac{1}{a} - \left(1 + \left(\frac{a}{z}\right)^2\right)^{1/2}\right]$$

$$-1 - \frac{a^2}{2z^2}$$

↳ taylor-expand

$$(1+x)^{1/2} \approx \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)$$

$$\to -\left[1 + \frac{a^2}{2z^2} - \frac{a^4}{8z^4}\right]$$

$$V = \frac{Qa}{4\pi\varepsilon_0 z^2}\left[\cancel{\frac{1}{a}}^{\,0} - 1 + \cancel{\frac{a^2}{2z^2}}^{\,0} - \cancel{\frac{a^4}{8z^4}}^{\,0}\right] \to z >> a$$

$$\boxed{V = \frac{-Qa}{4\pi\varepsilon_0 z^2}}$$

This represents the pontential of a ~~disk~~ specific charge ~~density~~ of charge

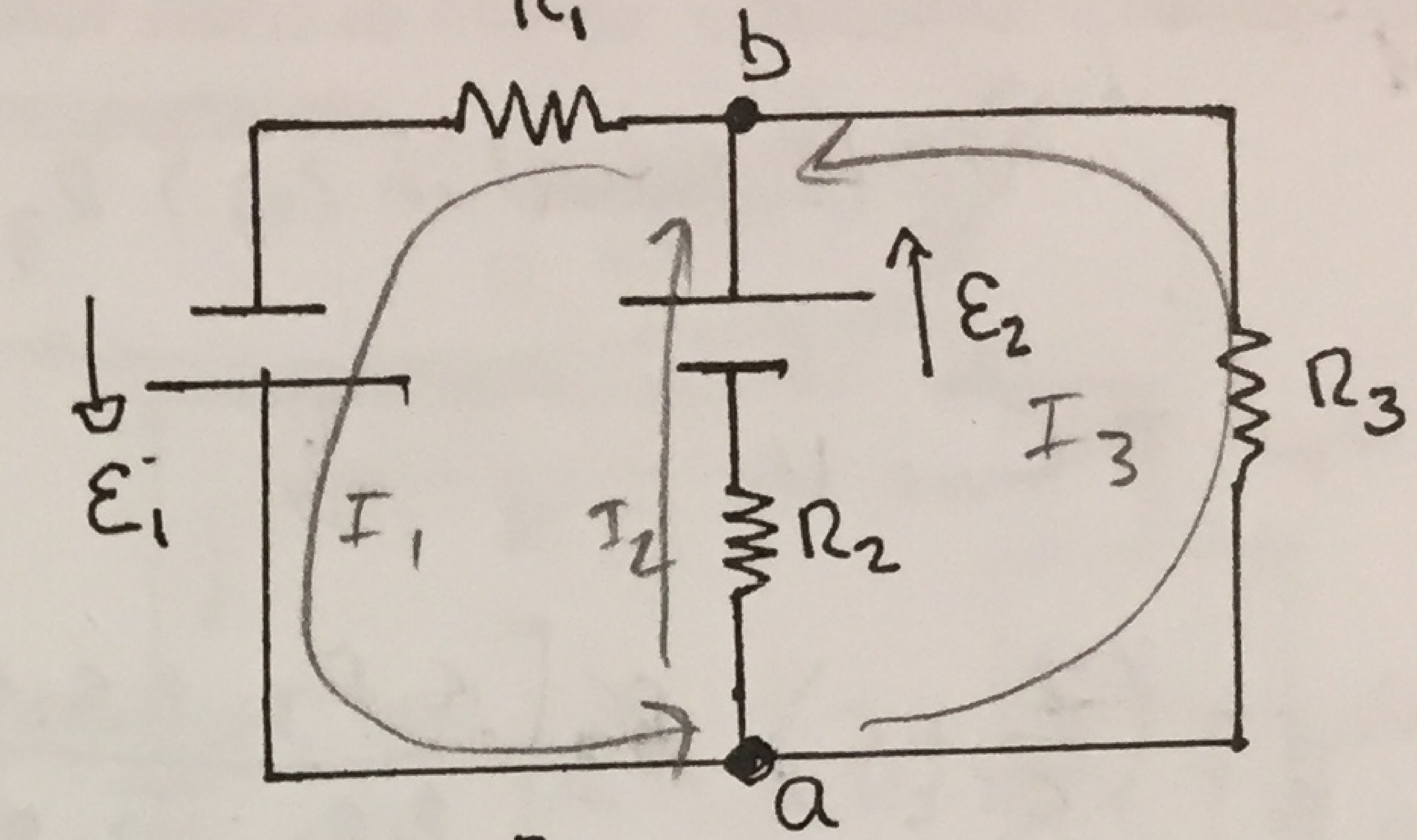
(27 Pts)

6. In the circuit shown, R_1, R_2, and R_3 are resistors, and the emf's ε_1 *and* ε_2 are produced by batteries.

(13) a. Show that the current that flows from a to b is given by

11

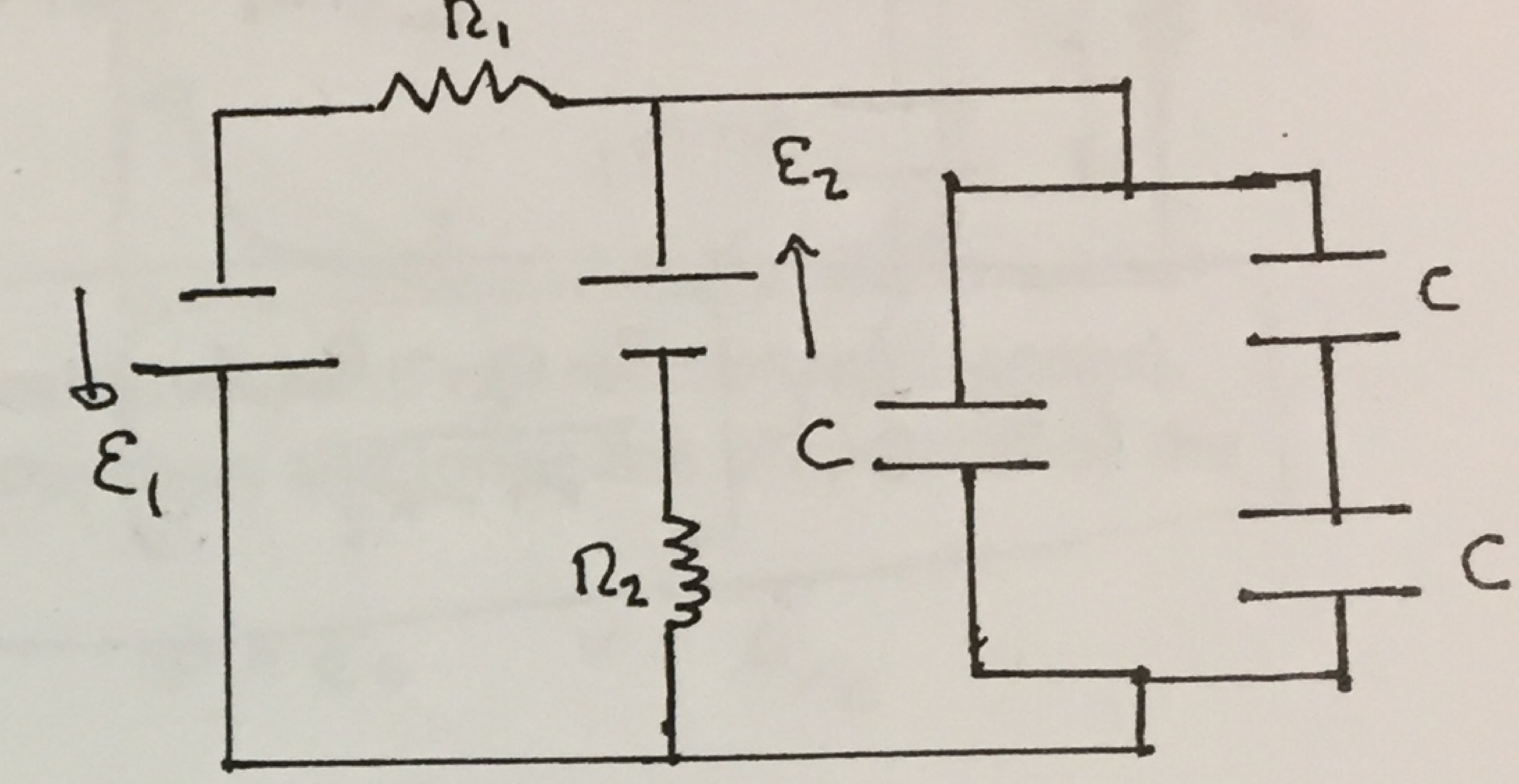
$$I = \frac{\varepsilon_1 R_3 + \varepsilon_2 (R_1 + R_3)}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$



(14) b. The resistor R_3 is replaced by the shown configuration of three equal capacitors C. Show that the total charge that is stored on the three capacitors is given by

2

$$q = \frac{3}{2} C \frac{R_1\varepsilon_2 - R_2\varepsilon_1}{R_1 + R_2}$$



a) $I_1 = I_2 + I_3$

$0 = \varepsilon_1 - I_2 R_2 + \varepsilon_2 - I_1 R_1$

want I_2

$0 = \varepsilon_2 + I_3 R_3 - I_2 R_2$

$0 = \varepsilon_1 - I_3 R_3 - I_1 R_1$

$\boxed{I_2 = I_1 - I_3}$ $\quad I_1 = \left(\frac{\varepsilon_1 - I_2 R_2 + \varepsilon_2}{R_1}\right)\frac{R_3}{R_3} \quad I_3 = \left(\frac{-\varepsilon_2 + I_2 R_2}{R_3}\right)\frac{R_1}{R_1}$

$I_1 - I_3 = \frac{R_3(\varepsilon_1 - I_2 R_2 + \varepsilon_2)}{R_1 R_3} - \frac{R_1(\varepsilon_2 + I_2 R_2)}{R_1 R_3}$

$I_2 = \frac{R_3 \varepsilon_1 - I_2 R_2 R_3 + R_3 \varepsilon_2 - R_1 \varepsilon_2 - R_1 I_2 R}{R_1 R_3}$

$I_2 + I_2 (R_2 R_3) + I_2 (R_1 R_2) =$

to denominator

$$\boxed{I_2 = \frac{\varepsilon_1 R_3 + \varepsilon_2 (R_1 + R_3)}{R_1 R_2 + R_1 R_3 + R_2 R_3}}$$

b) on back

5) total capacitance: $\frac{1}{C_{tot}} = \frac{1}{C} + \frac{1}{C} = \frac{2}{C} + C = \boxed{\frac{2}{C} + C}$ ← total capacitance

Voltage drop: $V = IR$

$V = (I \text{ ~~found~~ in (a)})\, R_3$

$Q_{tot} = C_{tot} V$

$$Q_{tot} = \left(\frac{2}{C} + C\right) \cancel{R_3} \left[\frac{\mathcal{E}_1 R_3 + \mathcal{E}_2 (R_1 + R_3)}{R_1 R_2 + R_1 \cancel{R_3} + R_2 \cancel{R_3}}\right]$$

$$\boxed{\mathcal{E} = \frac{3}{2} C \left[\frac{R_1 \mathcal{E}_2 - R_2 \mathcal{E}_1}{R_1 + R_2}\right]}$$

LOL

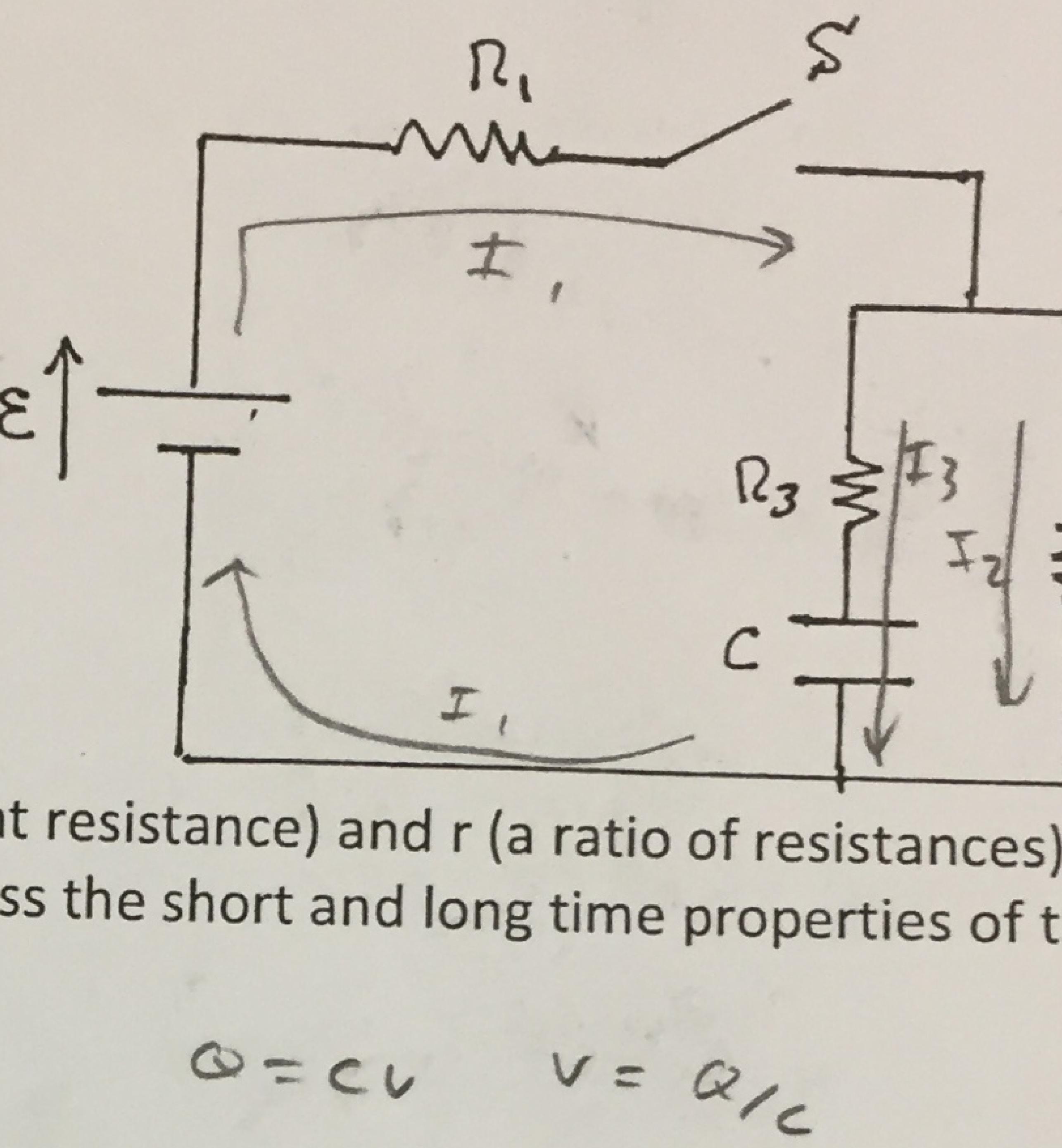
(30 Pts)
7. In the circuit shown, R_1, R_2, a…
…t = 0, the switc…

(0 Pts)

7. In the circuit shown, R_1, R_2, and R_3 are resistors, C is a capacitor, ε is a battery, and S is a switch. Before $t = 0$, the switch is open, and the capacitor is uncharged. At $t = 0$, the switch is closed, currents start to flow, and the capacitor starts to charge. Label the currents that flow through R_1, R_2, and R_3 as I_1, I_2, and I_3, respectively.

(10) a. Write the equations that result from applying Kirchhoff's First and Second Laws to this circuit.

10



(10) b. Prove that the charge q (t) on the capacitor obeys a first order differential equation of the form

0

$$\frac{dq}{dt} + \frac{q}{R_e C} = \frac{\varepsilon r}{R_e}$$

where you must find R_e (an equivalent resistance) and r (a ratio of resistances).

(10) c. Solve the equation for q(t), and discuss the short and long time properties of the charge and the flow of currents.

6

a) $\sum V_{loop} = 0$ $\quad \sum I_{junc} = 0$ $\qquad Q = CV \quad V = Q/C$

$$\boxed{\begin{aligned} I_1 &= I_2 + I_3 \\ 0 &= \varepsilon - I_1 R_1 - I_3 R_3 - Q/C \\ 0 &= \varepsilon - I_1 R_1 - I_2 R_2 \\ 0 &= -I_3 R_3 - Q/C + I_2 R_2 \end{aligned}}$$

b) R_e = total resistance

$\sum V = 0 \rightarrow 0 = \varepsilon - I R_e - Q/C$

$I = \frac{dQ}{dt}$, $\varepsilon \rightarrow \varepsilon r$ effective voltage felt depends on a ratio between the resistances, because the current splits.

$0 = \varepsilon r - \frac{dQ}{dt} R_e - Q/C$) divide by Re

$0 = \frac{\varepsilon r}{R_e} - \frac{dQ}{dt} - \frac{Q}{R_e C}$

$$\boxed{\therefore \frac{dQ}{dt} + \frac{Q}{R_e C} = \frac{\varepsilon r}{R_e}}$$

c) $\frac{dQ}{dt} = \frac{\varepsilon r}{R_e} - \frac{Q}{RC}$

$\frac{dQ}{dt} = \frac{1}{RC}\left[\varepsilon r C - Q\right]$

$\int \frac{dQ}{\varepsilon r C - Q} = \int \frac{dt}{RC}$

$\ln(\varepsilon r C - Q) = -\frac{t}{RC}$

$K e^{-t/RC} = \varepsilon r C - Q$

$Q = \varepsilon r C - K e^{-t/RC} \Rightarrow \boxed{Q_{(t)} = \varepsilon r C\left[1 - e^{-t/RC}\right]}$

when $t = 0$, $e^{-t/RC}$ goes to 1, so $Q(t) = 0$, as current must also be 0 before the switch is closed. when $t = \infty$, $e^{-t/RC} = 0$, so the charge become $\varepsilon r C$, and the current through that branch stops

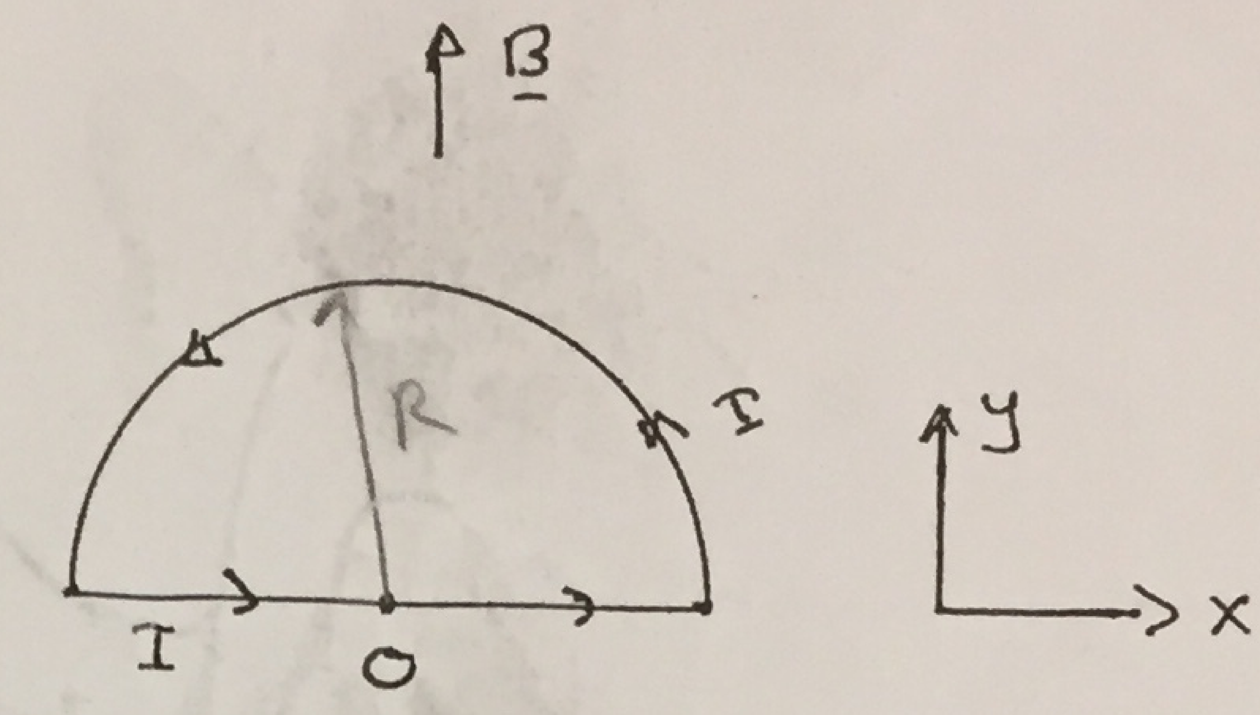
25 Pts)

8. A semi-circular wire loop with radius R carries a steady current I as shown. The loop lies in the x-y plane, and a uniform magnetic field $\underline{B}$ is in the y-direction as shown.

25

(12) a. By explicit calculation of the forces acting on the straight and curved parts of the loop, prove that the total force on the loop vanishes.

(13) b. By explicit calculation starting from $\underline{d\tau} = \underline{r} \times \underline{dF}$, find the direction and magnitude of the total torque about the center O of the loop, and confirm that the total toque is $\underline{\tau} = \underline{\mu} \times \underline{B}$.



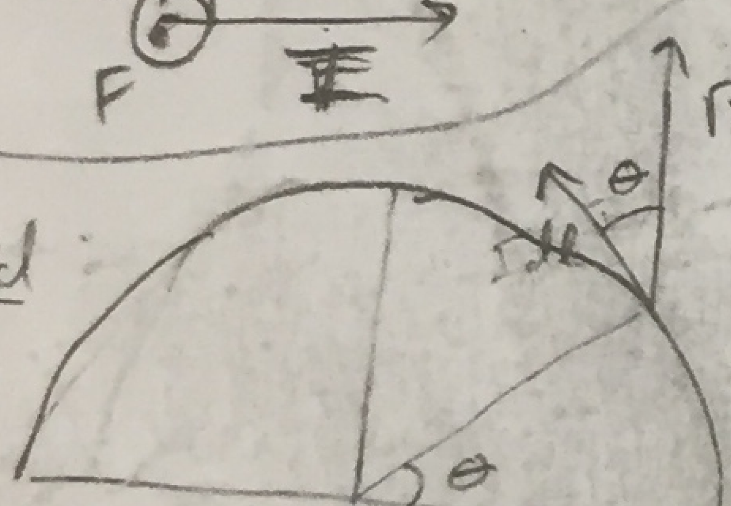
R, I

a) Claim: $F_{straight} = F_{curved}$

$dF_{straight} = I\,dl \times B = I\,dl\,B, \quad 2\int_0^R I\,dl\,B = \boxed{2IRB}$ ⊙ (up out of the page)

B, F, I

F_{curved}:



$dF = I\,dl \times B$

$dF_{in} = I\,dl\,B\sin\theta$

into page

$dl = R\,d\theta$

$dF_{in} = I\,R\,d\theta\,B\sin\theta$

$\int_0^\pi dF_{in} = IRB\int_0^\pi \sin\theta\,d\theta = IRB\,(\cos)\big|_0^\pi = \boxed{-2IRB}$

$F_{tot} = F_{straight} + F_{curved}$

$F_{tot} = 2IRB - 2IRB$

$\boxed{F_{tot} = 0}$

b) $d\tau = r \times dF$

$dF_{straight} = 2I\,dl\,B$ ⊙

$dF_{curved} = IR\,d\theta\,B\sin\theta$ ⊗

$d\tau_{straight} = r \times 2I\,dl\,B = 2IB(r \times dl) = 0$

$d\tau_{straight} = 0$

angle between r and dl is 180°, so $r \times dl = 0$

$d\tau_{curved} = r \times dF$

$= r \times IRB\sin\theta\,d\theta$

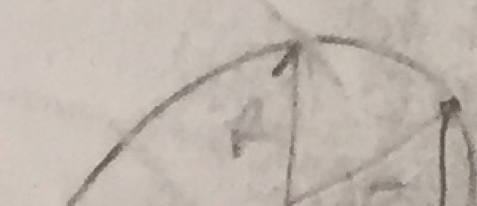
$d\tau_{curved} = (R)(IRB\sin\theta\,d\theta)\sin\theta$

$d\tau_{curved} = IR^2B\sin^2\theta\,d\theta$

$\tau_{curved} = \int_0^\pi IR^2B\sin^2\theta\,d\theta = IR^2B\int\sin^2\theta\,d\theta = IR^2B\left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right)$

Prove $\tau = \mu \times B$

$\tau_{total} = \tau_{curved} =$



$\left(\frac{\pi}{2}\right)IR^2B$

$\mu = (I \times area)\,\hat{n}$

$\mu = I\left(\frac{\pi r^2}{2}\right)$

$\tau = \left(\frac{\pi R^2}{2}\right)IB = \mu B$