33B Visan | Midterm 2 | February 20th

SOLUTIONS (UPDATED)

First Name: ____________________ ID#____________________

Last Name: ____________________

Section: ____________________ $= \begin{cases} 1a & \text{Tuesday with S. Kim} \\ 1b & \text{Thursday with S. Kim} \\ 1c & \text{Tuesday with J. Murphy} \\ 1d & \text{Thursday with J. Murphy} \\ 1e & \text{Tuesday with F. Robinson} \\ 1f & \text{Thursday with F. Robinson} \end{cases}$

Rules:

- There are **FOUR** problems for a total of 40 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	~~4~~	$\sum$

OMIT

(1) (10 points)
Find the general solution to the equation

$$x'' - 2x' - 3x = 3te^{2t}.$$

HOMOGENEOUS PART:

$$x''-2x'-3x=0 \Longrightarrow \lambda^2-2\lambda-3=0 \Longrightarrow (\lambda-3)(\lambda+1)=0$$

$$\lambda=-1,\ \lambda=3$$

$$\Longrightarrow x_h(t)=C_1e^{-t}+C_2e^{3t}$$

$g(t)=3te^{2t}$, GUESS $x_p(t)=(At+B)e^{2t} = (B)e^{2t}+(At)e^{2t}$

$$\Longrightarrow x_p'=(A+2B)e^{2t}+(2At)e^{2t}$$

$$\Longrightarrow x_p''=(4A+4B)e^{2t}+(4At)e^{2t}$$

So $x_p''-2x_p'-3x_p=\left[4A+4B-2(A+2B)-3(B)\right]e^{2t}+\left[4A-2(2A)-3(A)\right]te^{2t}$

$$=\left[2A-3B\right]e^{2t}+\left[-3A\right]te^{2t} = 3te^{2t}$$

$$\Longrightarrow A=-1,\ B=\frac{-2}{3}$$

So $\boxed{x(t)=C_1e^{-t}+C_2e^{3t}+\left(-t-\frac{2}{3}\right)e^{2t}}$

$$g = \frac{4\ln t}{t}$$

(2) (10 points)
Consider the equation

$$t^2x'' - tx' + x = \underline{4t\ln(t)} \quad \text{for} \quad t > 0.$$

(a) Verify that $\phi_1(t) = t$ and $\phi_2(t) = t\ln(t)$ form a fundamental set of solutions to the corresponding homogeneous equation for $t \in (0, \infty)$.

(b) Find a particular solution to the given inhomogeneous equation.

(c) Write down the general solution to the equation.

$$\left.\begin{array}{l} \phi_1 = t \\ \phi_1' = 1 \\ \phi_1'' = 0 \end{array}\right\} \Rightarrow t^2(0) - t(1) + (t) = 0 \ \checkmark$$

(a)

$$\left.\begin{array}{l} \phi_2 = t\ln t \\ \phi_2' = \ln t + 1 \\ \phi_2'' = \frac{1}{t} \end{array}\right\} \Rightarrow t^2\left(\frac{1}{t}\right) - t(\ln t + 1) + (t\ln t) = 0 \ \checkmark$$

USE VARIATION OF PARAMETERS: $x_p = V_1\phi_1 + V_2\phi_2$, NOTE $\boxed{W(t) = t\ln t + t - t\ln t = t}$

$$V_1 = -\int \frac{t\ln t \cdot 4\ln t}{t^2}\,dt = -\int \frac{4(\ln t)^2}{t}\,dt$$

$$\underline{V_1 = -\tfrac{4}{3}(\ln t)^3}$$

$\boxed{\text{so } x_p = -\tfrac{4}{3}(\ln t)^3 t + 2(\ln t)^2 t\ln t}$

$\boxed{x_p = \tfrac{2}{3}(\ln t)^3 t}$ (b)

$$V_2 = \int \frac{4t\ln t}{t^2}\,dt = \int \frac{4\ln t}{t}\,dt = \underline{2(\ln t)^2 = V_2}$$

$\boxed{(c): x(t) = c_1 t + c_2 t(\ln t) + \tfrac{2}{3}(\ln t)^3 t}$

(3) (10 points)

Find the solution to the following initial-value problem

$$x'' - 8x' + 17x = 0 \quad \text{with} \quad x(0) = 4 \quad \text{and} \quad x'(0) = -1.$$

$\lambda^2 - 8\lambda + 17 = 0 \quad \lambda = \dfrac{8 \pm \sqrt{64 - 68}}{2} = \dfrac{8 \pm 2i}{2} = 4 \pm i$

$\Rightarrow x(t) = e^{4t}\left[A_1 \cos t + A_2 \sin t\right]$

$x'(t) = 4e^{4t}\left[A_1 \cos t + A_2 \sin t\right] + e^{4t}\left[-A_1 \sin t + A_2 \cos t\right]$

$x(0) = 4 = A_1$

$x'(0) = -1 = 4A_1 + A_2 \Longrightarrow A_2 = -17$

$\boxed{x(t) = e^{4t}\left(4 \cos t - 17 \sin t\right)}$