33B Visan Midterm 1 January 30th

SOLUTIONS

First Name: ____________ ID#____________

Last Name: ____________

Section: ____________ $= \begin{cases} 1a & \text{Tuesday with S. Kim} \\ 1b & \text{Thursday with S. Kim} \\ 1c & \text{Tuesday with J. Murphy} \\ 1d & \text{Thursday with J. Murphy} \\ 1e & \text{Tuesday with F. Robinson} \\ 1f & \text{Thursday with F. Robinson} \end{cases}$

Rules:

- There are **FIVE** problems.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	5	$\sum$

(1)

Consider the differential equation

$$\frac{dx}{dt} = -(t + \cos(t))x^2.$$

(a) Find the general solution to this equation.
(b) Find the solution to this equation that satisfies the initial condition $x(0) = 1$.
(c) What is the interval of existence of the solution you found in part (b)?
(d) Find the solution to this equation that satisfies the initial condition $x(0) = 0$.

a) $$\int \frac{dx}{-x^2} = \int (t+\cos t)\,dt$$

$$\frac{1}{x} = \frac{1}{2}t^2 + \sin t + C \Longrightarrow \boxed{x(t) = \frac{1}{\frac{1}{2}t^2 + \sin t + C}}$$

b) $x(0) = \frac{1}{C} = 1 \Rightarrow C = 1$, so $\boxed{x(t) = \frac{1}{\frac{1}{2}t^2 + \sin t + 1}}$

c) NOTICE THAT $\sin t + 1 \geq 0$, AND $\frac{1}{2}t^2 > 0$ AS LONG AS $t \neq 0$. WE SEE $x(0)$ EXISTS BECAUSE $x(0) = 1$. FOR $t \neq 0$, $\frac{1}{2}t^2 + \sin t + 1 > 0$ SO $x(t)$ EXISTS EVERYWHERE. THE DIFF. EQ. ALSO EXISTS EVERYWHERE AND SO THE INTERVAL OF EXISTENCE IS $(-\infty, \infty)$

d) IF $x(0) = 0$, CAN'T USE GENERAL SOL...

BUT $x(t) = 0$ SATISFIES $x(0) = 0$ AND $\left.\begin{array}{l}\frac{dx}{dt} = 0 \\ -(t + \cos t)x^2 = 0\end{array}\right\}$ MATCH ✓

SO $\boxed{x(t) = 0}$ IS THE PARTICULAR SOLUTION

(2)

(a) Use ~~variation of parameters~~ to solve the following initial value problem

$$t\frac{dx}{dt} + x = 2t \quad \text{with} \quad x(1) = 0.$$

(b) Determine the interval of existence and provide a sketch of the solution.

a) $x' + \frac{1}{t}x = 2 \qquad u(t) = e^{\int \frac{1}{t}dt} = e^{\ln|t|}$ USE $u(t) = t$

$tx' + x = 2t$

$(tx)' = 2t$

$tx = t^2 + C$

$x = t + \frac{C}{t}$

$x(1) = 1 + C = 0 \Longrightarrow C = -1$

$\longrightarrow \boxed{x(t) = t - \frac{1}{t}}$

b) NEED $t \neq 0$, NEED $t=1$ INCLUDED, SO $\boxed{\text{INT OF EX. IS } (0, \infty)}$

(3)

(a) Find the value of the constant k such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$

$$y^3 + kxy^4 - 2x + (3xy^2 + 20x^2y^3)\frac{dy}{dx} = 0.$$

(b) Solve the equation using the value of k you obtained in part (a).

a) $\underbrace{(y^3 + kxy^4 - 2x)}_{P}\,dx + \underbrace{(3xy^2 + 20x^2y^3)}_{Q}\,dy = 0$

NESD $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$3y^2 + 40xy^3 = 3y^2 + 4kxy^3$ So NESD $\boxed{k=10}$

b) $F(x,y) = \int P\,dx$

$= \int (y^3 + 10xy^4 - 2x)\,dx = y^3x + 5x^2y^4 - x^2 + \phi(y)$

$\frac{\partial F}{\partial y} = 3y^2x + 20x^2y^3 + \phi'(y) = Q = 3xy^2 + 20x^2y^3$

$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = 0$

So solution is $F(x,y) = C$

$$\boxed{y^3x + 5x^2y^4 - x^2 = C}$$

(4)

A large tank is filled with 500 gallons of pure water. Brine containing 2 lb of salt per gallon is pumped into the tank at the rate of 5 gallons per minute. The well-mixed solution is pumped out at the same rate.

(a) Find the number of pounds of salt $x(t)$ in the tank at any time. Provide a sketch of the solution.

(b) What is the limiting value of $x(t)$ as $t \to \infty$?

$V = 500$

$$\frac{dQ}{dt} = \text{RATE IN} - \text{RATE OUT}$$

$$= (5)(2) - (5)\cdot\frac{Q}{500}$$

$$\frac{dQ}{dt} + \frac{Q}{100} = 10 \qquad u(t) = e^{\int \frac{1}{100} dt} = e^{t/100}$$

$$e^{t/100} Q' + \frac{e^{t/100} Q}{100} = 10 e^{t/100}$$

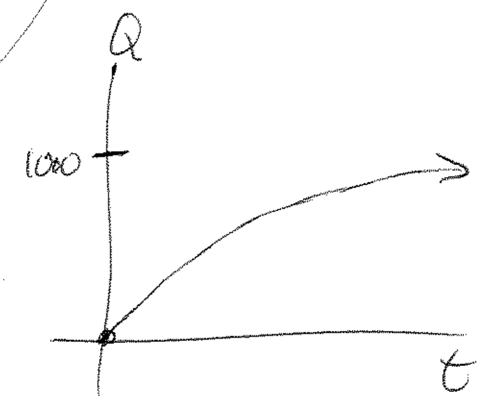
$$Q(0) = 0$$

$$\left(e^{t/100} Q\right)' = 10 e^{t/100}$$

$$\Rightarrow C = -1000$$

$$e^{t/100} Q = 1000 e^{t/100} + C$$

$$Q(t) = 1000 + C e^{t/100}$$



a) $\boxed{Q(t) = 1000 - 1000 e^{t/100}}$

b) $\boxed{\lim_{t \to \infty} Q(t) = 1000}$

(5)

Consider the differential equation

$$\frac{dx}{dt} = (x+1)(1-x^2).$$

(a) What is the largest rectangle in the tx plane on which you can apply the existence and uniqueness (Picard) theorem? Justify your answer.
(b) Identify the equilibrium points.
(c) Draw a phase diagram and identify the stable and unstable points.
(d) Sketch the equilibrium solutions in the tx plane. These equilibrium solutions divide the tx plane into regions. Sketch at least one solution curve in each of these regions.
(e) For the particular solution with initial condition $x(0) = 0.43$, what is the limit $\lim_{t\to\infty} x(t)$?

a) NO DISCONTINUITIES FOR EITHER f OR $\frac{\partial f}{\partial x}$

SO CAN APPLY OVER <u>ANY</u> RECTANGLE

(WOULD ACCEPT $(-\infty,\infty)\times(-\infty,\infty)$ OR "NO LARGEST RECTANGLE")