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Math 33B, Lecture 1
Winter 2017
02/27/17
Time Limit: 50 Minutes

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Signature:

Day \ T.A.	Madeleine	Alex	Thomas
Tuesday	2A	2C	2E
Thursday	2B	2D	2F

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and **cross the box** corresponding to your discussion section.
2. **Use a pen** to record your final answers.
3. Use the back of this page and the last pages if you need more space.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	20	20
2	18	18
3	18	16
4	14	12
Total:	70	66

1. (20 points) A spring hangs vertically from a hook. A 4 kg mass attached to the spring causes it to stretch ($\frac{4}{3} \cdot 9.8$)m. Suppose that the damping constant is 8 kg/sec. Let $x(t)$ be the displacement function in meters(m) form the mass-spring equilibrium after t seconds(sec).

   (a) Write a second order differential equation which is satisfied by x, and determine whether the motion is over-damped, under-damped, or critically damped.

   (b) Solve the system for the motion of the mass with $x(0) = 2$, $x'(0) = -5$, and sketch $x(t)$ when $t \geq 0$.

$m\ddot{x} + \mu x' + kx = 0 \Rightarrow$

$F = -kx$

$mg = -kx$

$4g = -k\left(\frac{4}{3}g\right)$

$\Rightarrow k = 3 \ \frac{N}{m}$

with $mx'' + \mu x' + kx = 0$,

a. $4x'' + 8x' + 3x = 0$

to explain the damping we can write in the alt. form $x'' + 2x' + \frac{3}{4}x = 0$. Looking back to the form $x'' + 2cx' + w^2 x = 0$, we see that $c = 1$. Because $c > w^2$, we have an **overdamped** system.

b. $4\lambda^2 + 8\lambda + 3 = 0$

$\lambda = \dfrac{-8 \pm \sqrt{64 - 48}}{8}$

$= \dfrac{-8 \pm \sqrt{16}}{8}$

$= \dfrac{-8 \pm 4}{8}$

$= -\dfrac{12}{8}$ or $\dfrac{-4}{8}$

$= -\dfrac{3}{2}$ or $-\dfrac{1}{2}$

So $x(t) = c_1 e^{-\frac{3t}{2}} + c_2 e^{-t/2}$

$x(0) = 2 \Rightarrow x(0) = c_1 e^0 + c_2 e^0$

$2 = c_1 + c_2$

$x'(t) = -\frac{3}{2} c_1 e^{-3t/2} - \frac{1}{2} c_2 e^{-t/2}$

$x'(0) = -\frac{3}{2} c_1 e^0 - \frac{1}{2} c_2 e^0 = -5$

$-5 = -\frac{3}{2} c_1 - \frac{1}{2} c_2$

$\Rightarrow 10 = 3c_1 + c_2$

$2 = c_1 + c_2$

$8 = 2c_1$

$c_1 = 4$

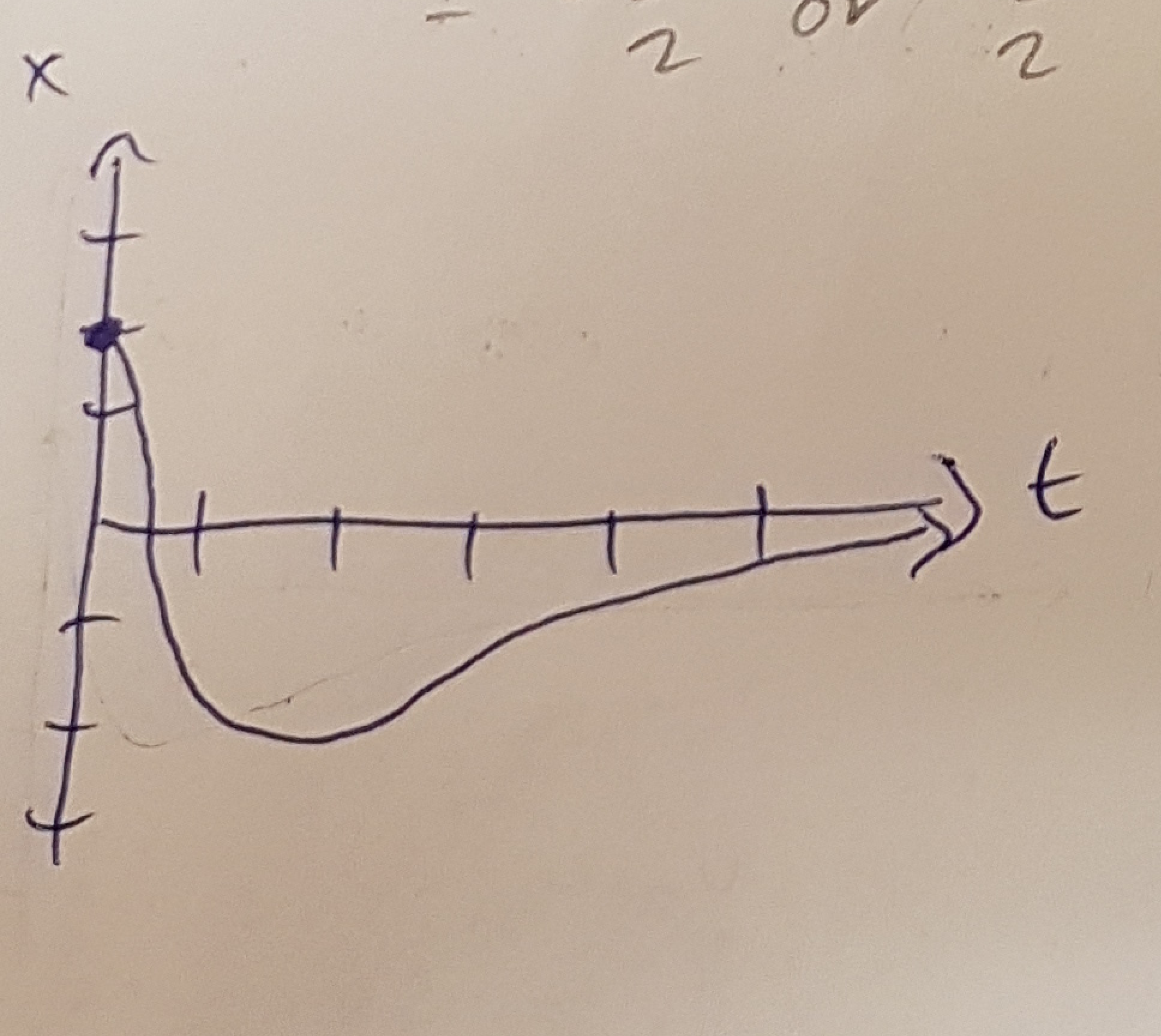
$2 = 4 + c_2$

$\Rightarrow c_2 = -2$

So

$x(t) = 4e^{-\frac{3t}{2}} - 2e^{-t/2}$

$x(t) = 4e^{-\frac{3t}{2}} - 2e^{-t/2}$



2. (18 points) Consider the following differential equation

$$t^2y'' + 4ty' + 2y = \frac{1}{t^2+1}, \quad \text{for } t > 0$$

(a) Find the general solution of the associated homogeneous equation.
(b) Find the general solution of the given inhomogeneous equation.

a. $t^2y'' + 4ty' + 2y = 0$

let $y = t^r$ $\quad y' = rt^{r-1}$ $\quad y'' = (r^2-r)t^{r-2}$

$\Rightarrow t^2(r^2-r)t^{r-2} + 4rt\,t^{r-1} + 2t^r = 0$

$\Rightarrow r^2 - r + 4r + 2 = 0$

$\Rightarrow r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -2 \quad r = -1$

So $y_1(t) = t^{-2} \quad y_2(t) = t^{-1}$

$y_h(t) = C_1 t^{-2} + C_2 t^{-1}$

$y_h(t) = C_1 t^{-2} + C_2 t^{-1}$

b. rewrite: $y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{1}{t^2(t^2+1)}$

var. of parameters look for $y_p = y_1v_1 + y_2v_2$

$\Rightarrow$ note: $y_1' = -2t^{-3} \quad y_2' = -t^{-2}$

So $y_1'v_1' + y_2'v_2' = \frac{1}{t^2(t^2+1)} \Rightarrow -2t^{-3}v_1' - t^{-2}v_2' = \frac{1}{t^2(t^2+1)} \Rightarrow -2v_1' - tv_2' = \frac{t}{t^2+1}$

$y_1v_1' + y_2v_2' = 0 \quad \Rightarrow t^{-2}v_1' + t^{-1}v_2' = 0 \quad \Rightarrow v_1' + tv_2' = 0$

$\Rightarrow v_1' + tv_2' = 0$

$-v_1' = \frac{t}{t^2+1}$

$-\frac{v_1'}{t} = v_2'$

$\Rightarrow v_1' = \frac{-t}{t^2+1}$

$\Rightarrow v_2' = \frac{1}{t^2+1}$

$\Rightarrow v_1(t) = -\int \frac{t}{t^2+1}\,dt \quad u = t^2+1 \quad du = 2\,dt$

$v_1(t) = -\frac{1}{2}\ln|t^2+1|$

$\Rightarrow v_2(t) = \int \frac{1}{t^2+1}\,dt$

$\Rightarrow v_2(t) = \tan^{-1}(t)$

So

$y_p = -\frac{1}{2}\ln|t^2+1|\,t^{-2} + \tan^{-1}(t)\frac{1}{t}$

So

$y(t) = -\frac{1}{2t^2}\ln|t^2+1| + \frac{1}{t}\tan^{-1}(t) + C_1t^{-2} + C_2t^{-1}$

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3. (18 points) Consider the differential equation

$$y'' - 4y' + 5y = 4\sin(t)$$

(a) Find the general solution of the associated homogeneous equation.

(b) Use the undetermined coefficients method to find the general solution of the given inhomogeneous equation.

(c) Consider the initial value problem which consists of the given equation together with the initial value conditions $y(0) = 0$, $y'(1) = 0$. Is it possible to apply the Uniqueness and Existence Theorem for second-order linear equations to this problem? Justify your answer.

a. $y'' - 4y' + 5y = 0$ so $y_1(t) = e^{2t}\cos(t)$

$\lambda^2 - 4\lambda + 5 = 0$ $y_2(t) = e^{2t}\sin(t)$

$\lambda = \frac{4 \pm \sqrt{16-20}}{2}$ and $y_h(t) = e^{2t}(C_1\cos t + C_2\sin t)$

$= \frac{4 \pm 2i}{2}$ $y_h(t) = e^{2t}(C_1\cos t + C_2\sin t)$

$= 2 \pm i$

b. assume $y_p(t) = a\cos t + b\sin t \Rightarrow y' = -a\sin t + b\cos t$

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$\Rightarrow y'' = -a\cos t - b\sin t$

so $-a\cos t - b\sin t + 4a\sin t - 4b\cos t + 5a\cos t + 5b\sin t = 4\sin t$

$\Rightarrow (-a - 4b + 5a)\cos t + (-b + 4a + 5b)\sin t = 4\sin t$

$\Rightarrow 4a - 4b = 0$ $\Rightarrow 4b + 4a = 4$

$\Rightarrow a = b$ $a + b = 4$

$a + a = 4 \Rightarrow a = 2$ $b = 2$

so $y_p(t) = 2(\cos t + \sin t)$

so $y(t) = 2(\cos t + \sin t) + e^{2t}(C_1\cos t + C_2\sin t)$

c. It is not. The theorem in question applies only when the initial values are given for the same t. Here we have $y(0)$ and $y'(1)$, so we cannot apply the theorem

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4. (14 points) Consider a linear homogeneous equation of the form

$$y'' + py' + qy = 0,$$

where p, q are constant real numbers. We are told that the complex number $\lambda = a + ib$ is a root of the equation's characteristic polynomial $x^2 + px + q$.

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(a) Explain why $\bar{\lambda} = a - ib$ is also a root of the characteristic polynomial.

(b) Explain why the functions $y_1(t) = e^{at}\cos(bt)$ and $y_2(t) = e^{at}\sin(bt)$ are solutions of the given equation.

a. we will first demonstrate some expressions by plugging in λ

$\lambda^2 + p\lambda + q = 0 \Rightarrow (a+ib)^2 + p(a+ib) + q = 0 \Rightarrow a^2 + 2abi - b^2 + pa + pib + q = 0$

$\Rightarrow (a^2 - b^2 + pa + q) + i(2ab + pb) = 0.$

which is zero iff $\Rightarrow a^2 - b^2 + pa + q = 0$ and $2ab + pb = 0$

then we see plugging in $\bar{\lambda}$ we get

$(a-ib)^2 + p(a-ib) + q = a^2 - 2abi + (-ib)^2 + pa - pib + q$

$= (a^2 - b^2 + pa + q) + i(-pb - 2ab) = (a^2 - b^2 + pa + q) - i(pb + 2ab)$ which equals

zero because we know from λ that $a^2 - b^2 + pa + q = 2ab + pb = 0$. Therefore, $\bar{\lambda}$ is a root.

b. $\lambda_1 = a + ib$ $\lambda_2 = a - ib$ $\Rightarrow y(t) = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$ solves, expanding

we get $y(t) = C_1 e^{at} e^{ibt} + C_2 e^{at} e^{-ibt}$

using Euler's Formula, $= e^{at}(C_1 \cos bt + iC_1 \sin(bt) + C_2 \cos(-ibt) + C_2 i \sin(-ibt))$

$= e^{at}(C_1 \cos(bt) + C_1 i \sin(bt) + C_2 \cos(bt) - C_2 i \sin(bt))$

we arrive at $y(t) = e^{at}((C_1 + C_2)\cos(bt) + i(C_1 - C_2)\sin(bt))$

of course i is just a constant itself (albeit a strange one), so we see the general sol ~~to be~~ $y(t) = e^{at}(A\cos bt + B\sin bt)$ why

general sol's are sums of linearly independent ~~solutions, so we have~~ A, B ~~can be~~ can be arbitrary real consts?

$y_1(t) = e^{at}\cos bt$, $y_2(t) = e^{at}\sin bt$