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Math 33B, Lecture 1
Winter 2016
02/03/17
Time Limit: 50 Minutes

Name (Print): River Robles

Signature:

Day \ T.A.	Madeleine	Alex	Thomas
Tuesday	~~2A~~	2C	2E
Thursday	2B	2D	2F

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and **cross the box** corresponding to your discussion section.
2. **Use a pen** to record your final answers.
3. Use the back of this page and the last page if you need more space.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	12	12
2	13	13
3	15	15
4	15	13
5	15	15
Total:	70	68

1. (12 points)

A tank is filled with 180 gallons of pure water. Solution containing 2.5 lb of salt per gallon is pumped into the tank at the rate of 2 gal/min. At the same time, the solution in the tank is pumped out at the same rate. Let $x(t)$ be the number of pounds of salt in the tank at time t.

(a) Use the information to write an initial value problem which is satisfied by $x(t)$.

(b) Find $x(t)$. $V_0 = 180$ $X_0 = 0$ $C_{in} = 2.5$ $V_{in} = 2$ $V_{out} = 2$

$C_{out} = \frac{x(t)}{V(t)} = \frac{x(t)}{180}$ $X' = X'_{in} - X'_{out} = C_{in} V'_{in} - C_{out} V'_{out} = 5 - 2 \cdot \frac{x}{180} = 5 - \frac{x}{90}$

a. $\boxed{\text{IVP: } x' = 5 - \frac{x}{90} \text{ w/ } x(0) = 0}$

b. $x' = \frac{5(90)}{90} - \frac{x}{90} = \frac{450 - x}{90}$

$\Rightarrow \frac{dx}{450 - x} = \frac{dt}{90}$

$\Rightarrow -\ln|450 - x| = \frac{t}{90} + C$

$x(0) = 0 \Rightarrow$

$-\ln|450| = 0 + C$

$\Rightarrow C = -\ln 450$

$\Rightarrow \ln|450 - x| = -\frac{t}{90} + \ln 450$

$\Rightarrow 450 - x = e^{-t/90 + \ln 450}$

$\Rightarrow 450 - x = 450e^{-t/90}$

$\Rightarrow \boxed{x(t) = 450(1 - e^{-t/90})}$

2. (13 points) Consider the differential equation

$$(e^{2x}y^2 - y)dx + (2e^{2x}y + e^x + 1)dy = 0$$

Find the general solution of the equation.

Confirming exact: $\frac{\partial}{\partial y}(e^{2x}y^2 - y) = 2e^{2x}y - 1$, $\frac{\partial}{\partial x}(2e^{2x}y + e^x + 1) = 4e^{2x}y + e^x$

not exact, must use integrating factors.

we want $\mu(x)$ s.t.

$$\frac{\partial}{\partial y}\mu(x)(e^{2x}y^2 - y) = \frac{\partial}{\partial x}\mu(x)(2e^{2x}y + e^x + 1)$$

$$\mu(x)(2e^{2x}y - 1) = \mu'(x)(2e^{2x}y + e^x + 1) + \mu(x)(4e^{2x}y + e^x)$$

$$\Rightarrow \mu'(x)(2e^{2x}y + e^x + 1) = \mu(x)(2e^{2x}y - 1 - 4e^{2x}y - e^x)$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{-(2e^{2x}y + e^x + 1)}{2e^{2x}y + e^x + 1}dx = -dx$$

$$\Rightarrow \ln|\mu| = -x + C \Rightarrow \text{let } C = 0 \Rightarrow \mu(x) = e^{-x}$$

multiply in,

$$(e^x y^2 - ye^{-x})dx + (2e^x y + 1 + e^{-x})dy = 0 \quad \text{is exact.}$$

$$F = \int e^x y^2 - ye^{-x}\,dx = e^x y^2 + e^{-x}y + \phi'(y)$$

$$F = \int 2e^x y + 1 + e^{-x}\,dy = e^x y^2 + ye^{-x} + y$$

so $\phi'(y) = y$, and

$$\boxed{e^x y^2 + ye^{-x} + y = C}$$

$$\boxed{e^x y^2 + ye^{-x} + y = C}$$

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3. (15 points) Consider the differential equation

$$(t^2-1)\frac{dx}{dt} + 2tx = te^t$$

(a) Solve the equation with the initial value condition $x(0) = -2$.

(b) What is the largest rectangle R in the tx-plane containing the point $(0, -2)$ to which we can apply the existence and uniqueness Theorem? Justify your answer.

a. $x' + \frac{2t}{t^2-1}x = \frac{te^t}{t^2-1}$

Solve by integrating factors: $a(t) = -\frac{2t}{t^2-1}$

$\Rightarrow \mu = e^{-\int -\frac{2t}{t^2-1}dt} = e^{\int \frac{2t}{t^2-1}dt} = e^{\int \frac{du}{u}} = e^{\ln|t^2-1|+C}$ $\quad u = t^2-1,\ du = 2t\,dt$

$= A(t^2-1)$

let $A=1$, then

$x(t)(t^2-1) = \int \frac{te^t}{t^2-1}(t^2-1)dt = \int te^t dt$ $\quad$ if $u = t$, $v = e^t$; $du = dt$, $dv = e^t dt$

we have $x(t)(t^2-1) = te^t - \int e^t dt = te^t - e^t + C$

$\Rightarrow x(t) = \frac{e^t(t-1)}{t^2-1} + \frac{C}{t^2-1}$

$(x(0) = -2) \Rightarrow -2 = \frac{e^0(-1)}{-1} + \frac{C}{(-1)} \Rightarrow -2 = 1 - C \Rightarrow C = 3$

so $\boxed{x(t) = \frac{e^t(t-1)}{t^2-1} + \frac{3}{t^2-1}}$ $\quad \boxed{x(t) = \frac{e^t(t-1)+3}{t^2-1}}$

b. existence th.;

$f(t,x) = \frac{1}{t^2-1}(te^t - 2tx)$

This is continuous everywhere but -1 and 1, so the existence th. applies only at the extent that

$\boxed{R = \{(t,x) \mid t \in (-1,1),\ x \in \mathbb{R}\}}$

uniqueness th.;

$\partial_x f = -\frac{2t}{t^2-1}$. This is also continuous everywhere but $t = 1$ and -1, so again the largest rectangle is

$\boxed{R = \{(t,x) \mid t \in (-1,1),\ x \in \mathbb{R}\}}$.

$\boxed{R = \{(t,x) \mid t \in (-1,1),\ x \in \mathbb{R}\}}$

4. (15 points) Consider the differential equation

$$\frac{dx}{dt} = x(x+3)^2 e^{-2x}$$

(a) Identify the equilibrium points and draw the phase line diagram of the equation.

(b) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points.

(c) Let $x(t)$ be a particular solution to the equation satisfying $x(0) = -1$. Is it possible that $\lim_{t\to\infty} x(t) = -2$? Justify your answer.

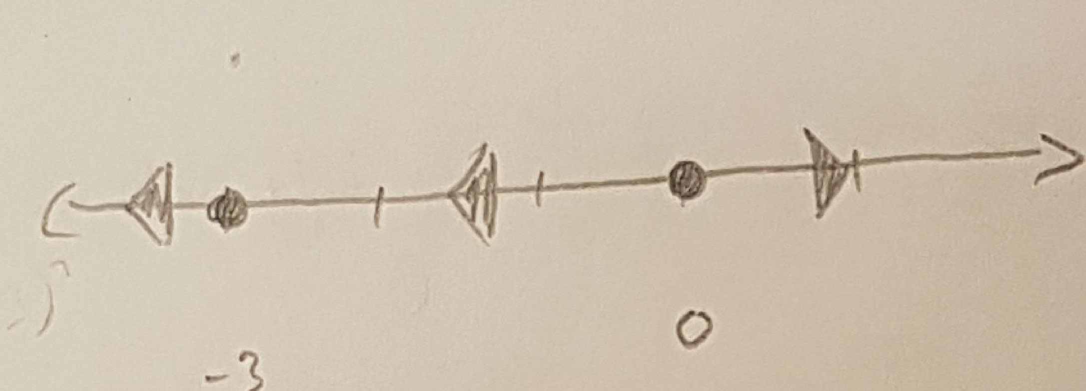
a. if $f = x(x+3)^2 e^{-2x}$, equib. points at

$\boxed{x = 0,\ x = 3}$

both are unstable

(see test to the right)

$f' = (x+3)^2 e^{-2x} + x[2(x+3)e^{-2x} - 2(x+3)^2 e^{-2x}]$

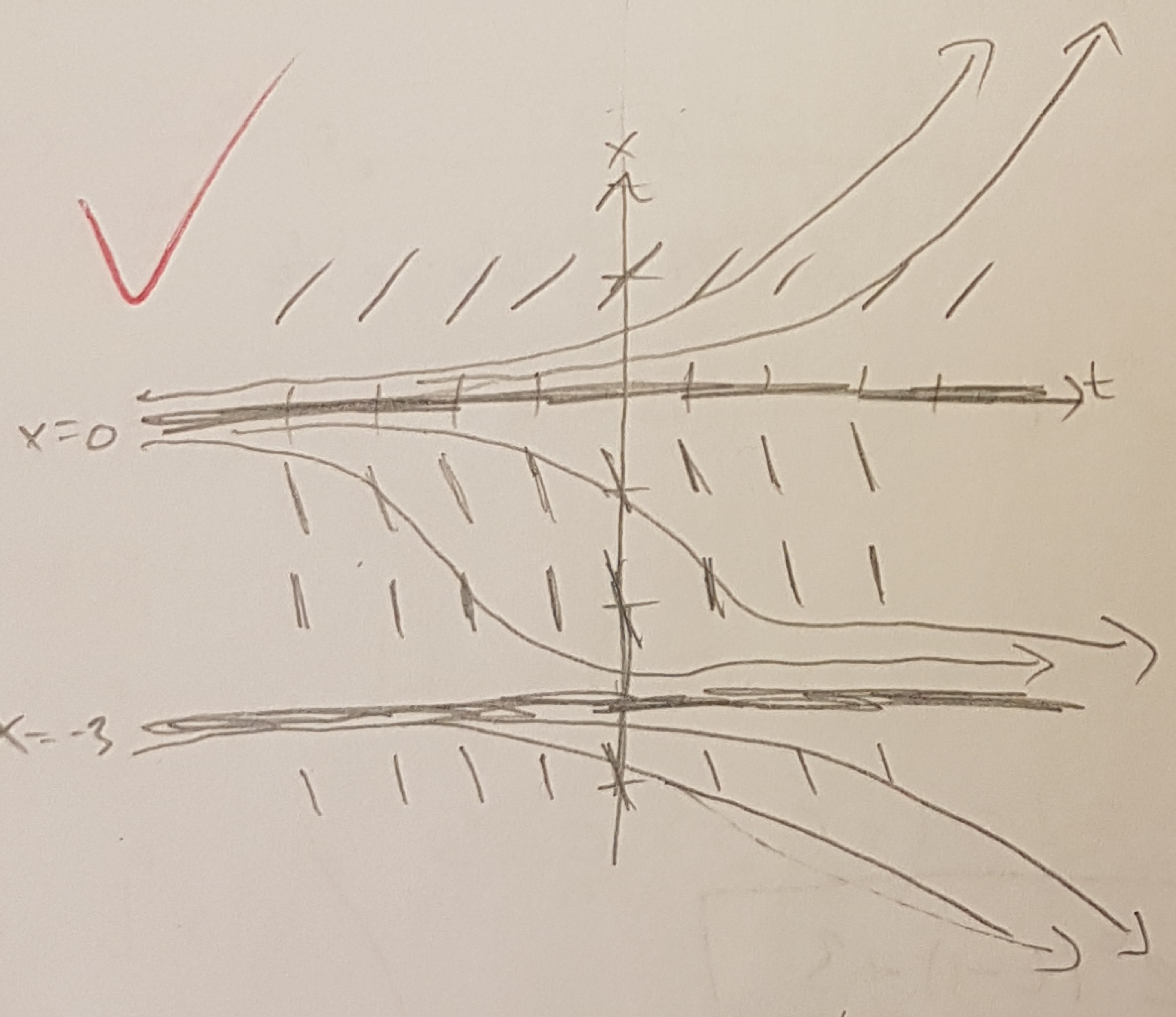


$f'(-3) = 36e^6 - 3(12e^6 - 72e^6)$

$\Rightarrow f'(-3) > 0$

$f'(0) = 9 + 0 > 0$

b.



c. no if $\lim_{t\to\infty} x(t) = 2$, that would imply that 2 was an equilibrium point. As graphically shown through the slope field above, any solution must be either dragged down to -3 or up to 0 if it begins in between the two.

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5. (15 points) Let $T(t)$ denote the temperature of an object after t minutes. Recall that according to Newton's law of cooling, T satisfies the following differential equation

$$\frac{dT}{dt} = -k(T - A),$$

where A is the (constant) ambient temperature, and k is some positive constant.

(a) Find the general solution of the equation.

(b) A cold beer at 40° is placed into a room. We are told that the beer's temperature after 10 minutes is 50°, and its temperature after 20 minutes is 52°. Find the temperature of the room.

a. $\frac{dT}{T-A} = -k\,dt \Rightarrow \ln|T-A| = -kt + C \Rightarrow T - A = e^{-kt+C} = Be^{-kt}$

$\Rightarrow \boxed{T = A + Be^{-kt}}$ where B is an arbitrary constant specific to the situation. $T = A$ is also a solution, so we say that B can be zero.

+7

b. $T = A + Be^{-kt}$

$T(0) = 40 \Rightarrow 40 = A + Be^{-0} = A + B$ (1) let $\alpha = e^{-10k}$

$T(10) = 50 \Rightarrow 50 = A + Be^{-10k}$ (2)

$T(20) = 52 \Rightarrow 52 = A + Be^{-20k}$ (3)

$40 = A + B$

$50 = A + B\alpha \rightarrow \frac{50-A}{B} = \alpha$

$52 = A + B\alpha^2$

$\Rightarrow 52 = A + B\frac{(50-A)^2}{B^2}$

$52 - A = \frac{(50-A)^2}{B}$

$\Rightarrow B = \frac{(50-A)^2}{52-A} = 40 - A$ from (1)

$\Rightarrow (50-A)^2 = (40-A)(52-A)$

$2500 - 100A + A^2 = 2080 - 92A + A^2$

$\Rightarrow 420 = 8A \Rightarrow \boxed{A = 52.5}$

$\boxed{A = 52.5°}$

$\boxed{A = 52.5°}$ +8

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