1. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & -2 \\ 1 & 4 & -2 \end{pmatrix}$$

$n \times m$

Recall that A corresponds to a linear transformation T_A.

(a) [2 pts] What are the domain and range of T_A?

YES

Domain: ~~ker(A)~~

$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 1 & 4 & -2 & 0 \end{array}\right) -(I)$

$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$1v_1 + 4x_2 - 2x_3 = 0$

$x_1 = -4x_2 + 2x_3$

$\boxed{\text{domain: } \mathbb{R}^m = \mathbb{R}^3}$

range: $\mathbb{R}^n$

$\boxed{\text{range: } \mathbb{R}^2}$

free variables

Declare: $x_2 = t$
$x_3 = r$

$y_1 = -4t + 2r$
$x_2 = t$
$x_3 = r$

domain: $\boxed{\text{span}\left(\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right)}$

$t\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

(b) [2 pts] Describe the image of T_A as a span of vector(s).

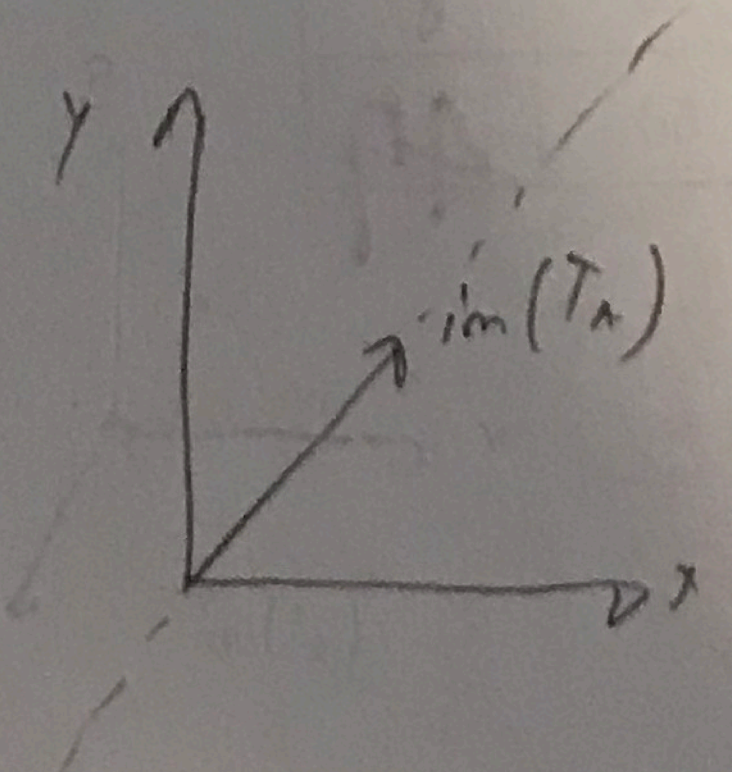
im T_A: span of columns of A

im T_A: $\text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}}_{\text{redundant}}\right)$

im T_A: $\text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

(c) [4 pts] Describe the image of T_A geometrically. Is it a line? A plane? Draw it.

The image T_A is a line in $\mathbb{R}^2$



6

2. [6 pts] Is the vector $\vec{b} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ a linear combination of the vectors $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$? If so, write down the linear combination in the format $\vec{b} = c_1\vec{v} + c_2\vec{w}$. If not, explain why not.

$A\vec{x} = \vec{b}$

$$\left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow{-\text{I}} \left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{array}\right) \xrightarrow[+2(\text{II})]{-3(\text{II})} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = 5 = c_1$

$x_2 = -3 = c_2$

$\vec{b} = c_1\vec{v} + c_2\vec{w}$

$$\begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

3. [5 pts] Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

Circle all of the following vectors which are members of ker(A).

ker A

$A\vec{x} = \vec{0}$

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \cancel{\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \quad \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

redundant form of $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ but still counts

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right) \begin{array}{l} \\ +(I) \\ \\ -3(I) \\ -(I) \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} +(IV) \\ \\ \\ \div -2 \\ +\frac{1}{2}(II) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} -2(IV) \\ \\ \\ \\ \\ \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_3 - x_4 = 0$ $\qquad x_1 = x_3 + x_4$ $\qquad$ Declare $x_3 = r$, $x_4 = t$

$x_2 + x_4 = 0$ $\qquad x_2 = -x_4$

$x_1 = r + t$

$x_2 = -t$

$x_3 = r$

$x_4 = t$

$$r\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\ker(A) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right)$$

4. Suppose you know that $A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}$.

6

(a) [6 pts] Find $(AB)^{-1}$. need to switch order

$(AB)^{-1} = B^{-1}A^{-1}$

$\begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 8 & 5 \end{pmatrix} = (AB)^{-1}$

6

(b) [6 pts] Find B.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$ac - bd = 5$

$\frac{1}{\det(B)} = \frac{1}{5}$

$B = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} \end{pmatrix}$

$\frac{1}{5}\begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}$

$B^{-1} = \frac{1}{\det B}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$B = 5\begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix}$

follows form

$B \rightarrow B^{-1}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$d = -1$
$-b = 1$
$-c = 3$
$a = 2$

$d = -1$
$b = -1$
$c = -3$
$a = 2$

$B = \begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix} \cdot \frac{1}{-5}$

$(B \mid I) \rightarrow (I \mid A)$

$B = A^{-1}$

2

(c) [2 pts] What was the rank of A? (this should require no computations)

$\left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right)^{-1}$

rank $A = 2$

$\left(\begin{array}{cc|cc} -1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 2 \end{array}\right) \xrightarrow{-3(2)} \left(\begin{array}{cc|cc} -1 & 0 & 1 & -1 \\ 3 & 1 & 0 & 5 \end{array}\right) \div 5$

$A\begin{pmatrix} -1 & 0 & 1 & -1 \\ \frac{3}{5} & \frac{1}{3} & 0 & 1 \end{pmatrix}$

5. (a) [2 pts] Write down the 2x2 matrix for rotation by an angle θ.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

7

(b) [2 pts] Use the determinant to show that this matrix is invertible.

$$\det(A) = \cos^2\theta - (-\sin^2\theta) \neq 0$$
$$= \cos^2 + \sin^2 \neq 0$$
$$1 \neq 0 \checkmark$$

(c) [3 pts] Explain geometrically what the inverse matrix should do, and write the inverse matrix down.

The inverse matrix will rotate by an angle of $-\theta$

rotate in opposite direction (so clockwise)

$$A^{-1} = \frac{1}{\underbrace{\cos^2\theta + \sin^2\theta}_{1}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A^{-1}$$

6. [4 pts] Suppose you know that $\vec{w}$ is in ker(B), and you also know that $B\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Use this information to find $B(3\vec{w} - 2\vec{v})$. $B\vec{w} = \vec{0}$

$3B\vec{w} - 2B\vec{v}$

def ker(B)

$B\vec{w} = \vec{0}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

4

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$

7. True or false (circle your answer; no justification needed). In all of the problems below, A is an nxn *square* matrix.

(a) [2 pts] If A is the coefficient matrix for some linear system, and rank$(A) = n$, then the system has a unique solution.

TRUE FALSE definition of invertibility

4

(b) [2 pts] If A is the coefficient matrix for some linear system, and rank$(A) < n$, then the system must have infinitely many solutions.

number of equations

TRUE FALSE

number of eq < number variables

free variable

(c) [2 pts] If A is the coefficient matrix for some linear system, and the system had a unique solution, then the RREF of A must be precisely the identity matrix $I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

TRUE FALSE