PRACTICE PROBLEMS FOR MATH 33A FINAL (the majority of these are questions I considered for the midterm, but couldn't fit):

1. Know definitions, and statements of theorems. Especially some of the big theorems (Rank-Nullity theorem, QR factorization, Spectral theorem, SVD factorization)
2. Computations of any of the stuff we discussed (ie first handful of problems from just about any section of the book, especially ch 5, ch 6, ch 7, ch 8).
3. If I have a basis $\mathfrak{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \cdots, \vec{\mathbf{v}}_n\}$ for $\mathbb{R}^n$, and I have a transformation T that sends each basis element to the difference of itself and the next one (wrapping around as follows)

$$T(\vec{\mathbf{v}}_1) = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$$
$$T(\vec{\mathbf{v}}_2) = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_3$$
$$\vdots$$
$$T(\vec{\mathbf{v}}_n) = \vec{\mathbf{v}}_n - \vec{\mathbf{v}}_1$$

   find the matrix B for the transformation T in $\mathfrak{B}$-coordinates, and then show that the sum $\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + \cdots + \vec{\mathbf{v}}_n$ is in the kernel. Is 1 an eigenvalue of B?
4. Suppose V is a subspace of $\mathbb{R}^5$, with basis $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$. Let $V^\perp$ be the orthogonal complement to V, having basis $\{\vec{\mathbf{v}}_4^\perp, \vec{\mathbf{v}}_5^\perp\}$. Let $T_V : \mathbb{R}^5 \to \mathbb{R}^5$ be the linear transformation of orthogonal projection onto V.

   If we put the two bases of V and $V^\perp$ together, we get a basis $\mathfrak{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4^\perp, \vec{\mathbf{v}}_5^\perp\}$ for $\mathbb{R}^5$. Find the matrix B for the linear transformation T_V in the coordinates of $\mathfrak{B}$.
5. Prove that if an $n \times n$ matrix has 1 as its only eigenvalue with algebraic *and* geometric multiplicity both equal to n, then the matrix must in fact be the identity matrix.
6. Give an example of a matrix that has 1 as its only eigenvalue, but that is not the identity matrix.
7. Give an example of a *real* 4x4 matrix whose eigenvalues are $i, -i, 1, -1$. It may be helpful to first decide on a 2x2 matrix with eigenvalues $\pm i$.
8. Let E_{λ_1} and E_{λ_2} be eigenspaces for two distinct eigenvalues $\lambda_1 \neq \lambda_2$. What is the intersection $E_{\lambda_1} \cap E_{\lambda_2}$?
9. Diagonalize $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and use this to find the diagonalization of A^4
10. Suppose A was an $n \times n$ matrix with n distinct eigenvalues all between 0 and 0.5. Use diagonalization to find the limit $\lim_{n\to\infty} A^n$.
11. Prove that, if A is similar to B, then the characteristic polynomials $f_A(\lambda)$ and $f_B(\lambda)$ are the same. *Hint:* $I = S^{-1}S = S^{-1}IS$
12. What is e^A for a matrix A? One possibility would be to try to define this using the Taylor series for e^x, plugging in the matrix A in place of x. Use this approach (look up the Taylor series for e^x if you've forgotten) and work out a simple formula for e^A in the case that A is diagonal. What if A isn't diagonal, but is diagonalizable?
13. Suppose A is an $n \times n$ symmetric matrix.

    Prove that $(A\vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \vec{\mathbf{v}} \cdot (A\vec{\mathbf{w}})$ for any two vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^n$.

    Prove that, if $\vec{\mathbf{v}} \perp \vec{\mathbf{w}}$ *and* $\vec{\mathbf{v}}$ is an eigenvector for A, then $A\vec{\mathbf{w}} \perp \vec{\mathbf{v}}$.
14. If A is a 10x12 matrix with rank 7, how many of its singular values must be zero?