Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

↳ $\dim(\operatorname{im}(A)) = 3$
$\dim(\ker(A)) = 3$

0

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

not possible – dimension of A is not high enough to have both rank & nullity = 3

4

Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

$(A\vec{x}) \cdot \vec{v} = \vec{0} \quad \forall\ \vec{x} \in \mathbb{R}$

unit vector parallel to $\vec{v}$: $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{1^2+1^2+1^2}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

$\vec{x} - (\vec{x} \cdot \vec{u})\vec{u} \perp \vec{v} \quad \forall\ \vec{x} \in \mathbb{R}$

$\vec{x} \cdot \vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3$

$(\vec{x} \cdot \vec{u})\vec{u} = \begin{bmatrix} \frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 \end{bmatrix}$

$\vec{x} - (\vec{x} \cdot \vec{u})\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 + \frac{1}{9}x_3 \end{bmatrix} = \begin{bmatrix} \frac{8}{9}x_1 - \frac{1}{9}x_2 - \frac{1}{9}x_3 \\ -\frac{1}{9}x_1 + \frac{8}{9}x_2 - \frac{1}{9}x_3 \\ -\frac{1}{9}x_1 - \frac{1}{9}x_2 + \frac{8}{9}x_3 \end{bmatrix}$

$A = \begin{bmatrix} \frac{8}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{8}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & \frac{8}{9} \end{bmatrix}$

$(A\vec{x}) \cdot \vec{v} = (\frac{8}{9}x_1 - \frac{1}{9}x_2 - \frac{1}{9}x_3)(1) + (-\frac{1}{9}x_1 + \frac{8}{9}x_2 - \frac{1}{9}x_3)(1) + (-\frac{1}{9}x_1 - \frac{1}{9}x_2 + \frac{8}{9}x_3)(1)$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ yikes

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

basis of im(A)

im(A) is the plane perpendicular to $\vec{v}$

$B = \{\vec{v}_1, \vec{v}_2\}$

$\vec{v}_1 =$

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

$\vec{0} \in V$?

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

2/3

V is a subspace if

1) for any $\vec{v} \in V$, $k\vec{v}$ also $\in V$ for any $k \in \mathbb{R}$

2) for any $\vec{v}, \vec{w} \in V$, $\vec{v}+\vec{w}$ also $\in V$

for $\vec{v}$ such that $\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}$, $\left.\begin{array}{l}(k\vec{v}) \cdot \vec{x} = k(\vec{v} \cdot \vec{x}) \\ (k\vec{v}) \cdot \vec{y} = k(\vec{v} \cdot \vec{y})\end{array}\right\}$ $\begin{array}{l}k(\vec{v} \cdot \vec{x}) = k(\vec{v} \cdot y) \\ (\vec{v} \cdot \vec{x}) = (\vec{v} \cdot y)\end{array}$

therefore $k\vec{v} \in V$

for $\vec{v}, \vec{w}$ s.t. $\begin{array}{l}\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y} \\ \vec{w} \cdot \vec{x} = \vec{w} \cdot \vec{y}\end{array}$, $\begin{array}{l}(\vec{v}+\vec{w}) \cdot \vec{x} = \vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x} \\ (\vec{v}+\vec{w}) \cdot \vec{y} = \vec{v} \cdot \vec{y} + \vec{w} \cdot \vec{y}\end{array}$

$\begin{array}{r}\bar{v} \cdot \bar{x} = \bar{v} \cdot \bar{y} \\ + \ \bar{w} \cdot x = \bar{w} \cdot \bar{y} \\ \hline (\bar{v} \cdot \bar{x}) + (\bar{w} \cdot \bar{x}) = (\bar{v} \cdot \bar{y}) + (\bar{w} \cdot \bar{y})\end{array}$

$(\vec{v}+\vec{w}) \in V$

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

$\vec{v}_1 \cdot \vec{x} = 2+1-0+1 = 4 = \vec{v}_1 \cdot \vec{y} = 1+2+1+0 = 4$ ✓

$\vec{v}_2 \cdot \vec{x} = 0+1+0+0 = 1 = \vec{v}_2 \cdot \vec{y} = 0+2-1+0 = 1$ ✓

$\vec{v}_3 \cdot \vec{x} = 2-1+0-1 = 0 = \vec{v}_3 \cdot \vec{y} = 1-2+1+0 = 0$ ✓

3/3

$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

Che cu!

$\hookrightarrow$ span $\{v_1, v_2, v_3\} = V$

$\forall \vec{v} \in V, \quad \vec{v} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\}$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$2v_1 + v_2 + 0 + v_4 = v_1 + 2v_2 - v_3 + 0$

$v_1 - v_2 + v_3 + v_4 = 0$

$\cancel{0}\,1/5$

. . . ?

Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} -\frac{R_2}{3} \\ \frac{R_3}{-6} \end{array} \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2/3 & 1/3 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & -1/6 \end{array}\right]$$

$$R_3-R_2 \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2/3 & 1/3 & 0 \\ 0 & 0 & 0 & 7/6 & -1/3 & -1/6 \end{array}\right]$$

free variable - matrix is not invertible

✓

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$ [There could be none.]

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -2/3 \\ 0 & 1 & 2 & -2/3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 - 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -1+\frac{8}{3} = 5/3 \\ 0 & 1 & 2 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x - 8z = \frac{5}{3}$

$y - 2z = -2/3$

Not final form

infinite solutions (free variable) of this form

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Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

$\vec{p} \perp P \qquad \vec{p} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{u}_p = \frac{\vec{p}}{\|\vec{p}\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$\vec{q} \parallel P \qquad \vec{q} = \vec{v} - (\vec{v} \cdot \vec{u}_p)\vec{u}_p$

$=$

$T(\vec{x}) = A\vec{x} \qquad A =$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.