Math 33A
Spring 2017
Midterm Exam 1
4/24/2017
Time Limit: 50 Minutes

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Section: 1B

This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Print your name legibly where requested on the top of this page, and print your initials on the top of every page, in case the pages become separated.

You should show your work clearly and concisely. If you need more space, use the back of the pages; clearly indicate when you have done this.

Draw a box around your final answer for each problem.

You may *not* use your books, notes, or any calculator on this exam.

Do not write in the table to the right.

Problem	Points	Score
1	30	30
2	25	25
3	25	25
4	20	20
Total:	100	100

RR

1. Let A be the matrix $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -4 & -2 & 4 \\ 3 & 1 & 1 & -3 \\ -1 & -1 & 0 & 3 \end{bmatrix}$.

(a) (20 points) Compute the reduced row echelon form (rref) of A.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -4 & -2 & 4 \\ 3 & 1 & 1 & -3 \\ -1 & -1 & 0 & 3 \end{bmatrix} \rightarrow \begin{array}{c} \\ \\ III-3I \\ IV+I \end{array} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -4 & -2 & 4 \\ 0 & 1 & -5 & -12 \\ 0 & -1 & 2 & 6 \end{bmatrix} \xrightarrow{-II/4} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 1 & -5 & -12 \\ 0 & -1 & 2 & 6 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} \\ \\ III-II \\ IV+II \end{array} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & -\frac{11}{2} & -11 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix} \rightarrow III \cdot \left(\frac{-2}{11}\right) \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} I-2III \\ II-\frac{1}{2}III \\ \\ IV-\frac{5}{2}III \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\boxed{\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

(b) (5 points) What is the rank of A? How do you know this?

There are three pivots in RREF(A), so the rank is 3.

(c) (5 points) Find a set of vectors that spans the kernel of A.

we want $A\vec{x} = \vec{0}$, so $\cancel{Rx} = [\text{rref}(A) \mid \vec{0}]$

this is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

let $x_4 = t$, then

$x_1 = t$

$x_2 = 2t$

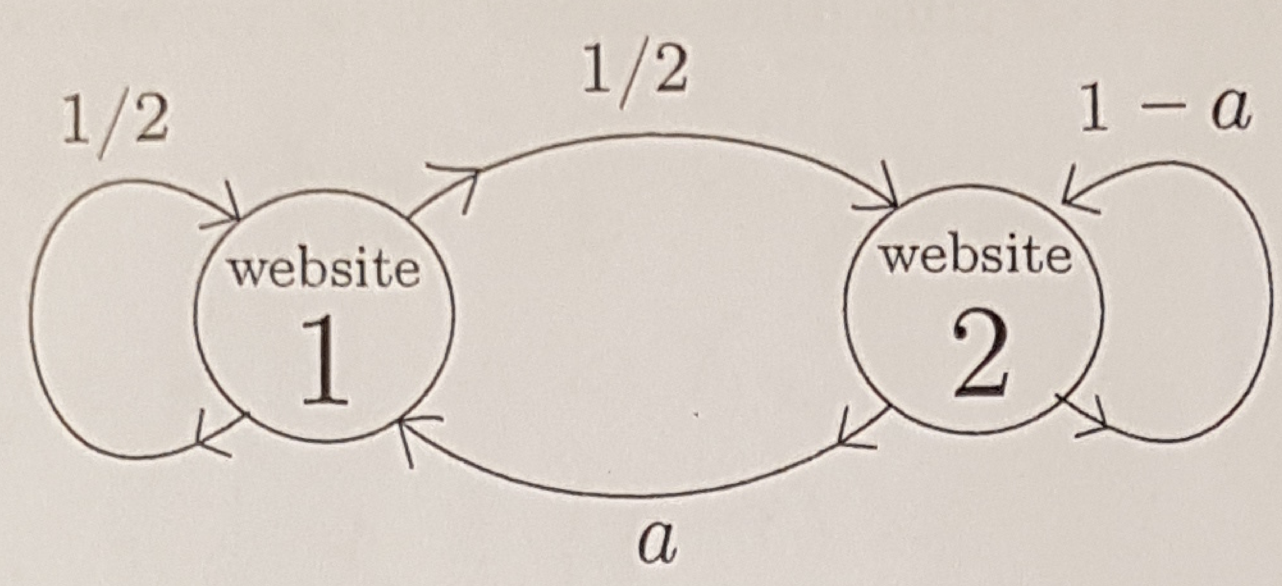
$x_3 = -2t$

so $\vec{x} = \begin{bmatrix} t \\ 2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$

$$\boxed{\text{So } \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \text{ spans } \ker(A)}$$

RR

2. Consider the mini-web shown below:



Unlike the examples you have seen on homework, this web has sites which link to themselves as well as to other sites. Also, each link has a weight between 0 and 1, such that the weights of all links leaving a site sum up to 1. The variable a is an undetermined weight with $0 \leq a \leq 1$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the distribution vector of people currently viewing each website; to be a distribution vector, $\vec{x}$ must satisfy $x_1 + x_2 = 1$. Assume that at each time step, everyone follows one of the links leaving their current website (or a self-link), with probabilities given by the weights on the links.

(a) (10 points) Write down the transition matrix A such that one time step has the effect of updating $\vec{x}$ to $A\vec{x}$. Give a few words of explanation. Your answer should depend on a.

$x_1' = \frac{1}{2}x_1 + a x_2$ $\qquad$ $x_2' = \frac{1}{2}x_1 + (1-a)x_2$

so $\boxed{A = \begin{bmatrix} \frac{1}{2} & a \\ \frac{1}{2} & (1-a) \end{bmatrix}}$

$\frac{1}{2}$ of viewers of x_1 return to x_1, a of those at x_2 go to x_1

likewise, $\frac{1}{2}$ of viewers of x_1 go to x_2 while the remainder of x_2 $(1-a)$ remain there.

10/10

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(b) (10 points) Find an equilibrium vector $\vec{x}_{\text{equ}}$ for A (this means $A\vec{x}_{\text{equ}} = \vec{x}_{\text{equ}}$; also, $\vec{x}_{\text{equ}}$ must be a distribution vector, so its entries must sum to 1.). Your answer should depend on a.

let $X_{eq} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then $x_1 = \frac{1}{2}x_1 + a x_2$

$x_2 = \frac{1}{2}x_1 + (1-a)x_2$

$\Rightarrow \frac{1}{2}x_1 - a x_2 = 0$ $\Rightarrow x_1 = 2a x_2$

$-\frac{1}{2}x_1 + a x_2 = 0$

let $x_2 = t$, then $x_1 = 2at$

we need $x_1 + x_2 = 1$, so $t(2a+1) = 1$

$\Rightarrow t = \frac{1}{2a+1}$

so $x_1 = \frac{2a}{2a+1}$ $x_2 = \frac{1}{2a+1}$

and $\boxed{\vec{X}_{eq} = \begin{bmatrix} \frac{2a}{2a+1} \\ \frac{1}{2a+1} \end{bmatrix}}$

10/10

(c) (5 points) For which value of a does site 2 have twice as many viewers as site 1 in the equilibrium state $\vec{x}_{\text{equ}}$?

want

$\frac{1}{2a+1} = 2\,\frac{2a}{2a+1}$

$1 = 4a$

$\boxed{a = \frac{1}{4}}$

5/5

$Ax=y$

RR

3. (a) (15 points) Let $B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Find a matrix A with $AB = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What is BA? You should not do another matrix multiplication to compute BA; instead, give the answer and a conceptual justification.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{c} \\ II-I \\ III-I \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \begin{array}{c} \\ II/2 \\ \frac{III}{2} \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \rightarrow \begin{array}{c} I+II \\ \\ III-II \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\rightarrow \begin{array}{c} I+III \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \rightarrow \boxed{A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

BA is also I_3 b/c A and B are square

15

RR

(b) (10 points) Let $C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$. Find a matrix A with $AC = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What is CA? Give a geometric interpretation of the transformation $T(\vec{x}) = CA\vec{x}$; be as specific as possible.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a_1 - a_2 = 1$ $\quad a_1 + a_2 = 0 \Rightarrow 2a_1 = 1 \rightarrow a_1 = \frac{1}{2} \Rightarrow a_2 = \frac{-1}{2}$

$b_1 - b_2 = 0 \quad b_1 + b_2 = 1 \Rightarrow 2b_1 = 1 \rightarrow b_1 = \frac{1}{2} \rightarrow b_2 = +\frac{1}{2}$

a_3, b_3 irrelevant

$$\boxed{A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}$$

$$CA = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$T(\vec{x})$ projects a vector $\vec{x} \in \mathbb{R}^3$ onto the x-y plane.
(ie deletes its z-component)

10

ΛR

5

4. Let Q be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is a rotation by $\frac{\pi}{2}$ (counter-clockwise).

(a) (5 points) What are the matrices of Q and Q^{-1}? In two separate sketches, draw the effect of Q and Q^{-1} on the unit square (the region $0 \leq x, y \leq 1$ in the xy plane).

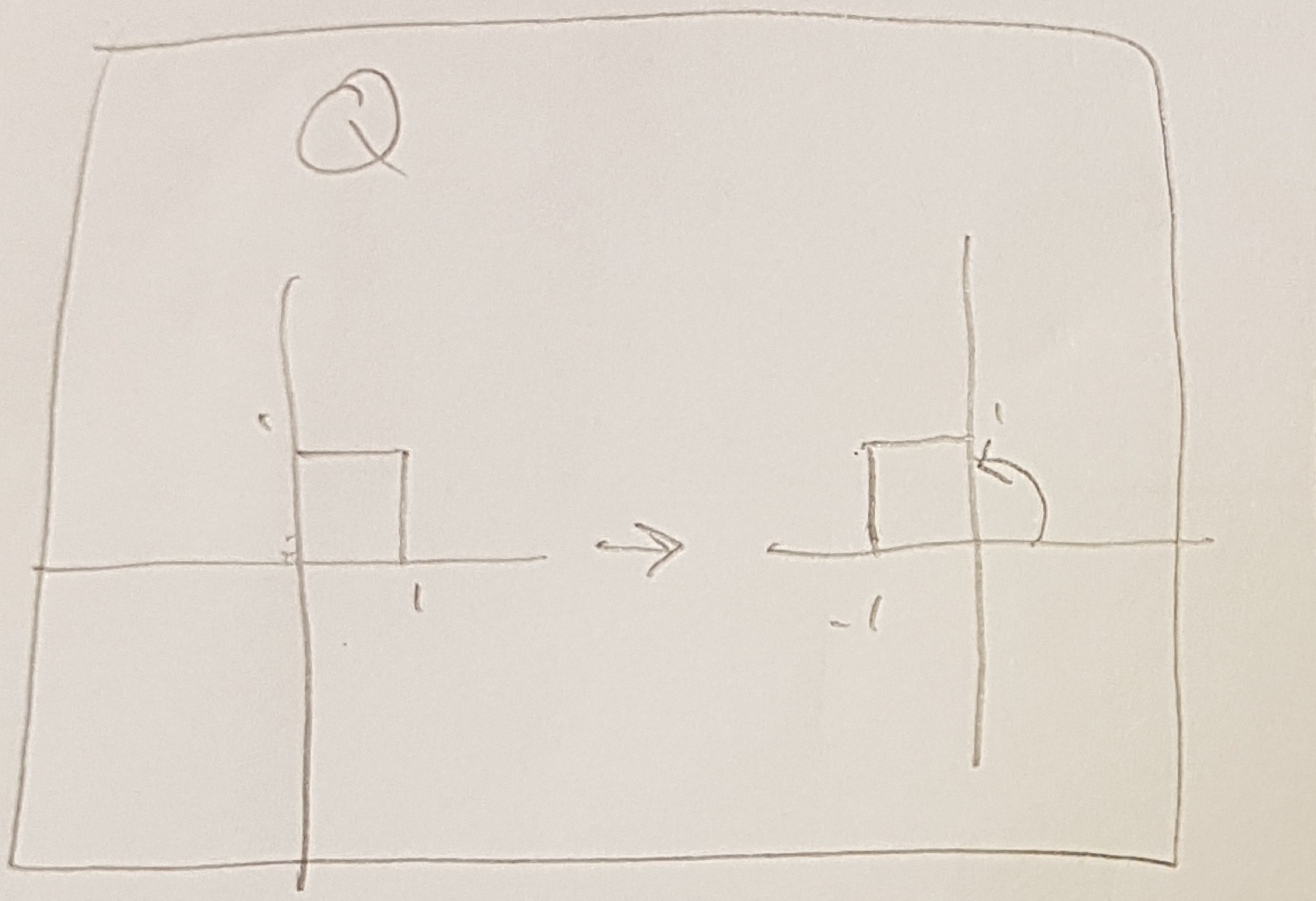
In general:

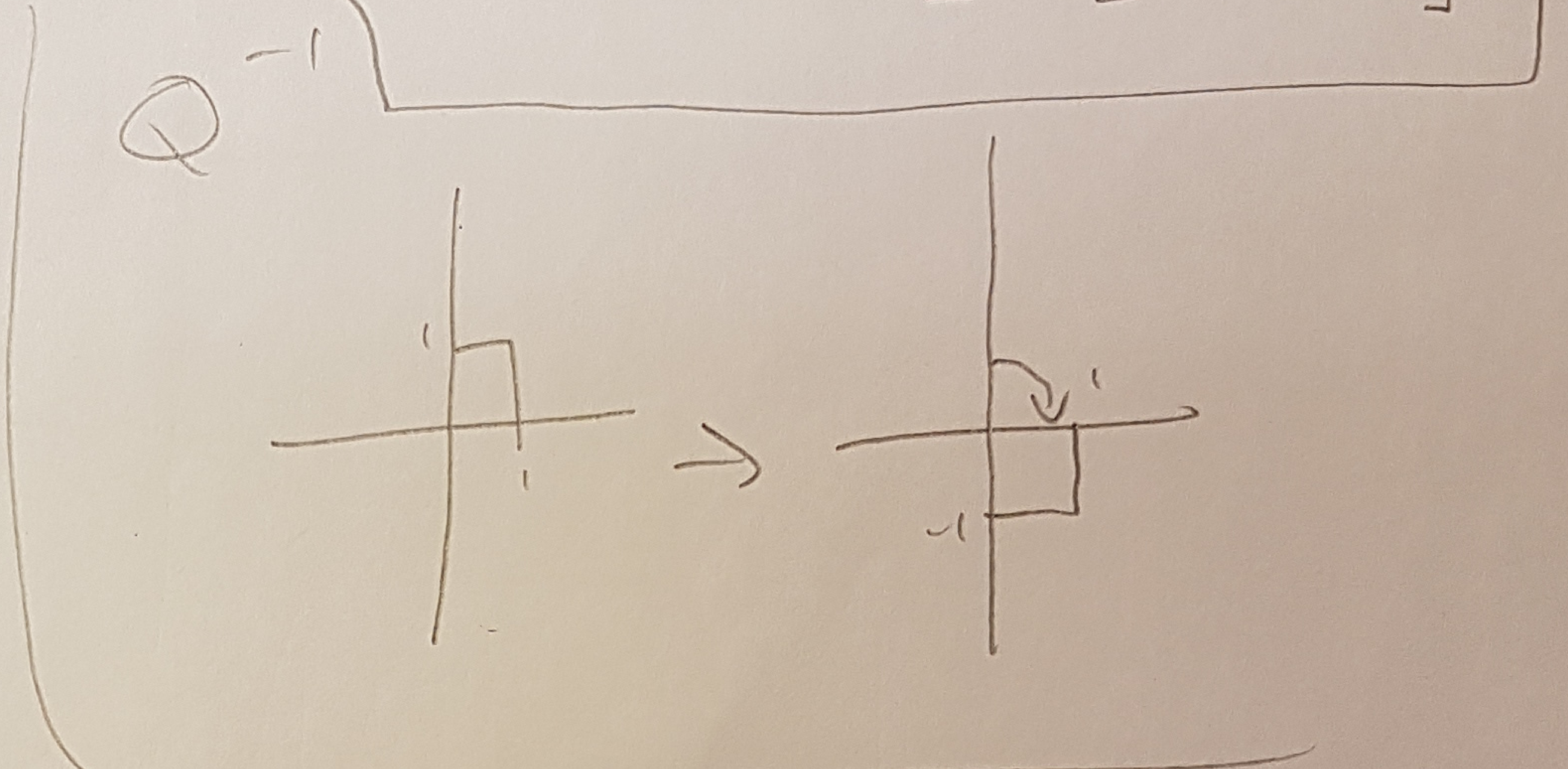
$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So

$$Q_{\frac{\pi}{2}} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Q^{-1} = Q_{-\frac{\pi}{2}} = \begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$





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(b) (5 points) For any real number a, let $T_a(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & a \end{bmatrix} \vec{x}$. What is the matrix of $Q \circ T_a \circ Q^{-1}$?

Your answer should depend on a.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -a & 0 \end{bmatrix} = \boxed{\begin{bmatrix} a & 0 \\ 0 & 2 \end{bmatrix}}$$

if $Q = Ax$ $T = Bx$ $Q^{-1} = A^{-1}x$

$$Q \circ T_a \circ Q^{-1}(x) = Q \circ T_a(A^{-1}x) = Q(BA^{-1}x) = ABA^{-1}x$$

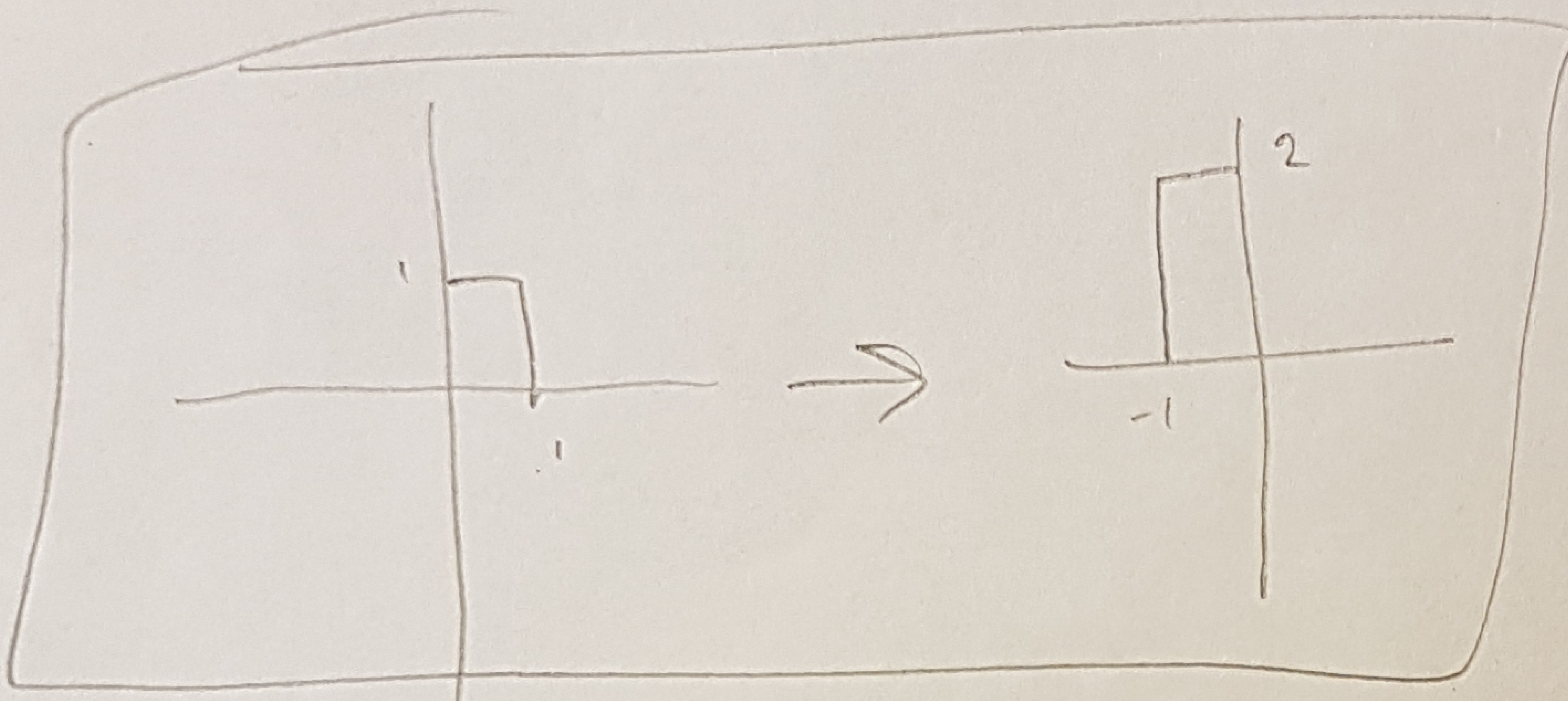
RR

5

(c) (5 points) Draw the effect of $Q \circ T_a \circ Q^{-1}$ on the unit square in the case $a = -1$.

let $U_a = Q \circ T_a \circ Q^{-1}$, then $U_{-1}(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

→ reflection over y-axis; vert. scale by 2



(d) (5 points) Let $\begin{bmatrix} x(a) \\ y(a) \end{bmatrix} = Q \circ T_a \circ Q^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. Give explicit formulas for $x(a)$ and $y(a)$ as functions of a, and draw the graphs of these functions. For which value(s) of a is $Q \circ T_a \circ Q^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ a scalar multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

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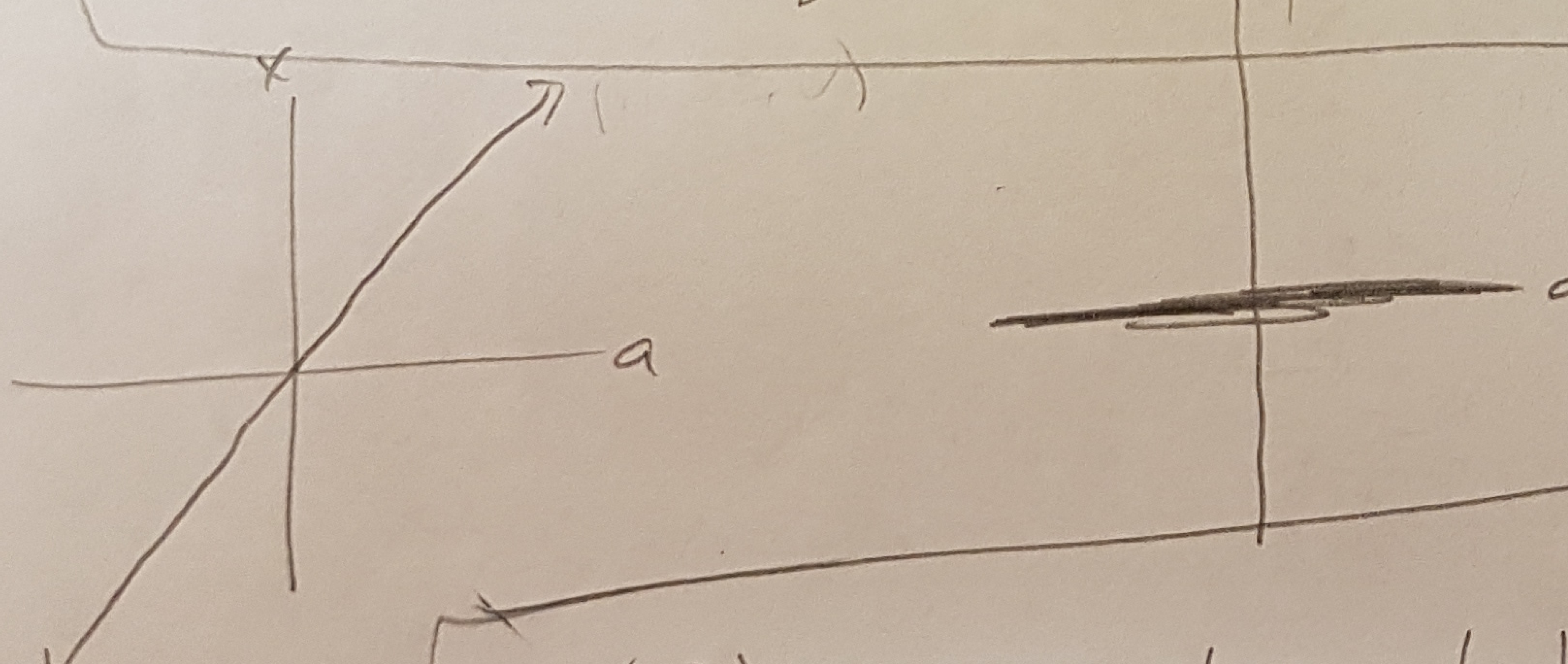
$U_a\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ → $x(a) = a$, $y(a) = 0$



$U_a(\vec{e}_1)$ is a scalar multiple of $\vec{e}_1$ $\forall a \in \mathbb{R}$