Gallauer Spring

Math 33A/1
Spring 2016
05/13/16
Time Limit: 50 Minutes

Nam
SID Number:

| Day \ T.A. | David | Casey | Adam |
|---|---|---|---|
| Tuesday | 1A | 1C | 1E |
| Thursday | 1B | 1D | 1F |

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This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | 10 |
| 2 | 10 | 10 |
| 3 | 10 | 9 |
| 4 | 10 | 10 |
| Total: | 40 | 39 |

1. (10 points) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by the vectors

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-2\\-1\\4\end{bmatrix}.$$

(a) Suppose the $\mathcal{B}$-coordinate vector of $\vec{x}$ is $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}0\\1\\2\end{bmatrix}$. What is $\vec{x}$?

(b) Find the $\mathcal{B}$-coordinate vector of $\vec{y} = \begin{bmatrix}1\\2\\8\end{bmatrix}$.

(c) Find the $\mathcal{B}$-matrix of $A = \begin{bmatrix}5 & -5 & 1\\-1 & 2 & 0\\-31 & 37 & -5\end{bmatrix}$.

a) $\vec{x} = \begin{bmatrix}1&1&-2\\1&1&-1\\1&2&4\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = 1\begin{bmatrix}1\\1\\2\end{bmatrix} + 2\begin{bmatrix}-2\\-1\\4\end{bmatrix} = \begin{bmatrix}1-4\\1-2\\2+8\end{bmatrix} = \begin{bmatrix}-3\\-1\\10\end{bmatrix}$

b) $\begin{bmatrix}1&1&-2\\1&1&-1\\1&2&4\end{bmatrix}[\vec{y}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\8\end{bmatrix}$

$\begin{bmatrix}1&1&-2&1\\1&1&-1&2\\1&2&4&8\end{bmatrix}\begin{matrix}\\-I\\-I\end{matrix} \to \begin{bmatrix}1&1&-2&1\\0&0&1&1\\0&1&6&7\end{bmatrix} \to \begin{bmatrix}1&1&-2&1\\0&1&6&7\\0&0&1&1\end{bmatrix}-II \to$

$\begin{bmatrix}1&0&-8&-6\\0&1&6&7\\0&0&1&1\end{bmatrix}\begin{matrix}+8III\\-6III\end{matrix} \to \begin{bmatrix}1&0&0&2\\0&1&0&1\\0&0&1&1\end{bmatrix}$ $\boxed{[\vec{y}]_{\mathcal{B}} = \begin{bmatrix}2\\1\\1\end{bmatrix}}$

c) $AS = SB$ ~~$A = SBS^{-1}$~~ $S = \begin{bmatrix}1&1&-2\\1&1&-1\\1&2&4\end{bmatrix}$ $S^{-1}$: $\begin{bmatrix}1&1&-2&1&0&0\\1&1&-1&0&1&0\\1&2&4&0&0&1\end{bmatrix}\begin{matrix}\\-I\\-I\end{matrix} \to$

$B = S^{-1}AS$ B = $\mathcal{B}$ matrix of A

$\begin{bmatrix}1&1&-2&1&0&0\\0&0&1&-1&1&0\\0&1&6&-1&0&1\end{bmatrix} \to \begin{bmatrix}1&1&-2&1&0&0\\0&1&6&-1&0&1\\0&0&1&-1&1&0\end{bmatrix}-II \to \begin{bmatrix}1&0&-8&2&0&-1\\0&1&6&-1&0&1\\0&0&1&-1&1&0\end{bmatrix}\begin{matrix}+8III\\-6III\end{matrix} \to \begin{bmatrix}1&0&0&-6&8&-1\\0&1&0&5&-6&1\\0&0&1&-1&1&0\end{bmatrix}$

$S^{-1} = \begin{bmatrix}-6&8&-1\\5&-6&1\\-1&1&0\end{bmatrix}$

$B = \begin{bmatrix}-6&8&-1\\5&-6&1\\-1&1&0\end{bmatrix}\left(\begin{bmatrix}5&-5&1\\-1&2&0\\-31&37&-5\end{bmatrix}\begin{bmatrix}1&1&-2\\1&1&-1\\1&2&4\end{bmatrix}\right) = \begin{bmatrix}-6&8&-1\\5&-6&1\\-1&1&0\end{bmatrix}\begin{bmatrix}1&2&-1\\1&1&0\\1&-4&5\end{bmatrix}$

$= \boxed{\begin{bmatrix}1&0&1\\0&0&0\\0&-1&1\end{bmatrix}}$

2. (10 points) (a) Find the QR-factorization of the matrix $M = \begin{bmatrix} 3 & -6 & 7 \\ 0 & -3 & 5 \\ 4 & -8 & 1 \end{bmatrix}$.

(b) Explain why $R \cdot M^{-1}$ is orthogonal.

a) $\vec{u}_1 = \frac{1}{\underbrace{\sqrt{3^2+4^2}}_{r_{11}}} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$

$\vec{v}_2^{\perp} = \begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} - \overbrace{\left( \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} \right)}^{r_{12}} \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} + 10 \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\underbrace{\sqrt{3^2}}_{r_{22}}} \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$\vec{v}_3^{\perp} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} - \overbrace{\left( \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \right)}^{r_{13}} \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} - \underbrace{\left( \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \right)}_{r_{23}} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$

$\vec{u}_3 = \frac{1}{\underbrace{\sqrt{25}}_{r_{33}}} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}$

$$\boxed{M = \overbrace{\begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & -1 & 0 \\ 4/5 & 0 & -3/5 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} 5 & -10 & 5 \\ 0 & 3 & -5 \\ 0 & 0 & 5 \end{bmatrix}}^{R}}$$

b) $M = QR$ ~~[illegible]~~ Since the column vectors of $Q$ form an orthonormal basis for $\mathbb{R}^3$ as shown by the Gram-Schmidt process in part (a), $Q$ is an orthogonal matrix. Therefore, $Q^{-1} = Q^T$ is also an orthogonal matrix. These come from the properties of orthogonal matrices.

$$M = QR$$
$$Q^{-1}M = R$$
$$Q^T M = R$$
$$Q^T = RM^{-1}$$

Since $Q^T$ is known to be orthogonal, $RM^{-1}$ is also orthogonal.

3. (10 points) Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -2 & -1 \\ -6 & 8 & 6 \\ 8 & -12 & -10 \end{bmatrix}.$$

(a) Find a basis for image$(A)^\perp$.

(b) Compute rank$(A)$.

(c) Find all $2 \times 2$ matrices which are both orthogonal and skew-symmetric.

4

a) $\text{image}(A)^\perp = \ker(A^T)$ $A^T = \begin{bmatrix} -2 & 2 & -6 & 8 \\ 2 & -2 & 8 & -12 \\ 1 & -1 & 6 & -10 \end{bmatrix} \div -2 \to \begin{bmatrix} 1 & -1 & 3 & -4 \\ 2 & -2 & 8 & -12 \\ 1 & -1 & 6 & -10 \end{bmatrix} \begin{matrix} \\ -2I \\ -I \end{matrix} \to \begin{bmatrix} 1 & -1 & 3 & -4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 3 & 6 \end{bmatrix} \div 2$

basis$(\text{image}(A)^\perp) = $ basis$(\ker(A^T))$

$\begin{bmatrix} 1 & -1 & 3 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix} \begin{matrix} -3II \\ \\ -3II \end{matrix} \to \begin{bmatrix} 1 & -1 & 0 & -10 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_2 = s$ $x_1 = 10t + s$ $x_4 = t$ $x_3 = -2t$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 10 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

basis$(\text{image}(A)^\perp) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

2

b) rank$(A)$ = rank$(A^T)$ rank$(A^T) = 4 - \dim(\ker(A^T))$

$= 4 - 2$

$= \boxed{2}$

3

c) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\sqrt{a^2 + c^2} = 1$

$a^2 + c^2 = 1$

$a = \cos\theta, c = \sin\theta$

A orthogonal, so $A^T = A^{-1}$

$b = \cos(\theta \pm \frac{\pi}{2}) = -\sin\theta$ or $\sin\theta$

$d = \sin(\theta \pm \frac{\pi}{2}) = \cos\theta$ $-\cos\theta$

Case 1: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $A^T = -A$ iff $\cos\theta = 0$ + $\sin\theta = 1$

Case 2: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ & $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ $A^T = A$ always, so never skew-symmetric

Case 2 yields none. back to case 1: if $\cos\theta = 0$, $\sin\theta = \pm 1$: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are only skew-symmetric orthogonal 2x2 matrices

4. (10 points) (a) Find the least-squares solution to the system

$$\left|\begin{aligned} x \quad &= 1 \\ -x - \ y &= 0 \\ x + 2y &= 0 \\ -x + \ y &= 2 \end{aligned}\right|$$

(b) Compute the error for the least-squares solution in (a).

(c) Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ be an orthonormal basis of $\mathbb{R}^n$. Find the least-squares solution to the system

$$A\vec{x} = \vec{u}_n, \qquad \text{where } A = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_{n-1} \\ | & | & & | \end{bmatrix}.$$

a) $\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}}_{\vec{b}}$ $A^TA = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$ $(A^TA)^{-1} = \frac{1}{24-4}\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 2/10 \end{bmatrix}$

5

$\begin{bmatrix} x \\ y \end{bmatrix} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 2/10 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 2/10 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}}$

b) $A\begin{bmatrix} x \\ y \end{bmatrix} - \vec{b} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ +1/2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \\ +1/2 \\ -1 \end{bmatrix}$

2

$\left\| A\begin{bmatrix} x \\ y \end{bmatrix} - \vec{b} \right\| = \cancel{\sqrt{(-\tfrac{3}{2})^2 + (\tfrac{3}{2})^2 + (-1)^2} = \sqrt{\tfrac{18}{4} + 1} = \sqrt{\tfrac{22}{4}} = \tfrac{\sqrt{22}}{2}}$

$\sqrt{\frac{9}{4} + \frac{1}{4} + 1} = \sqrt{\frac{14}{4}} = \boxed{\frac{\sqrt{14}}{2}} = \sqrt{\frac{7}{2}}$

c) $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$ $A^TA = I_n$ $(I_n)^{-1} = I_n$

$(A^TA)^{-1} = (I_n)^{-1} = I_n$

3

$= I_n A^T \vec{u}_n$

$I_{n-1}$.

$= A^T\vec{u}_n$

$= \cancel{[(\vec{u}_1 \cdot \vec{u}_n) + (\vec{u}_2 \cdot \vec{u}_n) + \cdots + (\vec{u}_n \cdot \vec{u}_n)]^T}$

$= [0 + 0 + \cdots + \vec{u}_{n-1}\vec{u}_n]^T$

$= \boxed{[0]}$