Gallauer Spring '16

Math 33A/1
Spring 2016
04/22/16
Time Limit: 50 Minutes

Name (Print):
SID Number:

Day \ T.A.	David	Casey	Adam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	10
2	10	10
3	10	7
4	10	10
Total:	40	37

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

OH

1. (10 points) Consider the following system of linear equations:

$$\begin{vmatrix} 2x - 4y + z = 0 \\ x + ky \quad = 0 \\ 2y + kz = 1 \end{vmatrix},$$

where k is a real constant.

(a) For which values of k does the system have a unique solution? No solutions? Infinitely many?

(b) Solve the system when $k = 0$.

a) $\cancel{\begin{bmatrix} 2 & -4 & 1 & 0 \\ 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & k & 0 & 0 \\ 2 & -4 & 1 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \xrightarrow{-2(I)} \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & -4-k & 1 & 0 \\ 0 & 2 & k & 1 \end{bmatrix}}$

$\begin{bmatrix} 2 & -4 & 1 & 0 \\ 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \div 2 \rightarrow \begin{bmatrix} 1 & -2 & \frac{1}{2} & 0 \\ 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} -I \rightarrow \begin{bmatrix} 1 & -2 & \frac{1}{2} & 0 \\ 0 & k+2 & -\frac{1}{2} & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 2 & k & 1 \\ 0 & k+2 & -\frac{1}{2} & 0 \end{bmatrix} \div 2 \rightarrow \begin{bmatrix} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{k}{2} & \frac{1}{2} \\ 0 & k+2 & -\frac{1}{2} & 0 \end{bmatrix}$

~~If $k=-2$, no solution one sol'n~~

~~if $k=0$, one sol'n~~

~~if $k \neq 0, 2$, this normal system that can be row-reduced to a unique solution~~

~~only way to get $0=$ Number is if $1-k-2 = \frac{k}{2}+\frac{1}{2}$~~

cont. on back

~~Unique solution for all $k \in \mathbb{R}$~~

Answer: $\boxed{\begin{array}{l}\text{No solutions for } k=-1 \\ \text{Unique solution for all } k \neq -1\end{array}}$

b) $\begin{bmatrix} 2 & -4 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} -2(I) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \div -4 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} -2(II) \rightarrow$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \cdot 2 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} +\frac{1}{4}(III) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \boxed{\begin{array}{l} x = 0 \\ y = \frac{1}{2} \\ z = 2 \end{array}}$

+2

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1a) $\begin{bmatrix} 1 & -2 & 1/2 & 0 \\ 0 & 1 & k/2 & 1/2 \\ 0 & k+2 & -1/2 & 0 \end{bmatrix} \begin{matrix} +2\,\mathrm{II} \\ \\ -(k+2)\,\mathrm{II} \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1+2k}{2} & 1 \\ 0 & 1 & k/2 & 1/2 \\ 0 & 0 & -\frac{1}{2}-(\frac{k}{2})(k+2) & -\frac{k+2}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1+2k}{2} & 1 \\ 0 & 1 & k/2 & 1/2 \\ 0 & 0 & \frac{-k^2-2k-1}{2} & -\frac{k+2}{2} \end{bmatrix}$

solve $-k^2-2k-1=0$

$-(k^2+2k+1)=0$

$-(k+1)^2=0$

$k=-1$

if $k=-1$, last row is $0 \quad 0 \quad 0 \quad 3/2$

no real solution

If $k \neq -1$, then last row is $0 \quad 0 \quad a \quad b$ for some a and b where $a \neq 0$.

This would create a matrix that can be row-reduced and solved to yield a unique solution.

so, answer on front (no sol'n for $k=-1$, unique sol'n for $k \neq -1$)

+8

OH

2. (10 points) (a) Find the inverse of the matrix $\begin{bmatrix}1&0&1\\2&1&2\\3&0&0\end{bmatrix}$.

(b) Find a 4×3 matrix A satisfying

$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\0\\4\end{bmatrix},\qquad A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix},\qquad A\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}.$$

(*Hint: Notice that e.g.* $\begin{bmatrix}1\\0\\0\end{bmatrix}=2\begin{bmatrix}1\\1\\1\end{bmatrix}-2\begin{bmatrix}0\\1\\0\end{bmatrix}-\begin{bmatrix}1\\0\\2\end{bmatrix}$.)

4

a) $\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\2&1&2&0&1&0\\3&0&0&0&0&1\end{array}\right]\begin{matrix}\\-2(\mathrm{I})\\-3(\mathrm{I})\end{matrix}\to\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-2&1&0\\0&0&-3&-3&0&1\end{array}\right]\div -3\to\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-2&1&0\\0&0&1&1&0&-\frac13\end{array}\right]-\mathrm{III}\to$

$\left[\begin{array}{ccc|ccc}1&0&0&0&0&\frac13\\0&1&0&-2&1&0\\0&0&1&1&0&-\frac13\end{array}\right]$ $\underbrace{}_{A^{-1}}$

$$A^{-1}=\begin{bmatrix}0&0&\frac13\\-2&1&0\\1&0&-\frac13\end{bmatrix}$$

6

b) $A\begin{bmatrix}0\\1\\0\end{bmatrix}=\vec v_2=\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix}$

bc $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is 2nd standard basis vector

$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\vec v_1+\vec v_2+\vec v_3=\begin{bmatrix}1\\2\\0\\4\end{bmatrix}$ $\qquad A\begin{bmatrix}1\\0\\2\end{bmatrix}=\vec v_1+2\vec v_3=\begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}$

$\vec v_1=\begin{bmatrix}1\\2\\0\\4\end{bmatrix}-\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix}-\vec v_3$

$\vec v_1=\begin{bmatrix}1\\1\\1\\6\end{bmatrix}-\vec v_3$ $\qquad \begin{bmatrix}1\\1\\1\\6\end{bmatrix}-\cancel{\vec v_3}+\cancel{2}\vec v_3=\begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}$

$\vec v_3=\begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\\6\end{bmatrix}$

$\vec v_3=\begin{bmatrix}-4\\1\\-2\\-5\end{bmatrix}$

$\vec v_1=\begin{bmatrix}1\\1\\1\\6\end{bmatrix}-\begin{bmatrix}-4\\1\\-2\\-5\end{bmatrix}$

$\vec v_1=\begin{bmatrix}5\\0\\3\\11\end{bmatrix}$

$A=\begin{bmatrix}|&|&|\\\vec v_1&\vec v_2&\vec v_3\\|&|&|\end{bmatrix}=\boxed{\begin{bmatrix}5&0&-4\\0&1&1\\3&-1&-2\\11&-2&-5\end{bmatrix}}$

Check: $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\0\\4\end{bmatrix}$ ✓

$A\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}$

OH

3. (10 points) (a) Find the matrix of reflection about the line $4y = 3x$ in $\mathbb{R}^2$.

(b) Describe the kernel of this matrix geometrically.

(c) Is the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $4y = 3x^2$ a subspace of $\mathbb{R}^2$? Justify your answer.

a) $4y = 3x$, $y = \frac{3}{4}x \rightarrow$ line spanned by $\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$, same as line spanned by $4\begin{bmatrix} 1 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = L_1$

-3

$\text{proj}_{L_1} \rightarrow A = \frac{1}{4^2+3^2}\begin{bmatrix} 4^2 & 4\cdot 3 \\ 4\cdot 3 & 3^2 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} \frac{16}{25} & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$

$\text{ref}_{L_1} \rightarrow B = A - I_2 = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -9/25 & 12/25 \\ 12/25 & -16/25 \end{bmatrix}}$

Not orthog proj.

b) The kernel of B is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Only the origin will collapse to zero when reflected about a line through the origin since any other vector will become its mirror image, and thus another nonzero vector. Thus, $\ker(B) = \{\vec{0}\}$ = the origin.

c) No. This set of vectors is not closed under scalar multiplication, as would be expected for a parabola. For example, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is in the set since $4\cdot 3 = 3(2^2)$, but $2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is not since $4\cdot 6 = 24$ but $3\cdot(4^2) = 48$, which does not satisfy $4y = 3x^2$. Thus this set of vectors is not a subspace of $\mathbb{R}^2$.

check $\begin{bmatrix} -9/25 & 12/25 \\ 12/25 & -16/25 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = [$

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4. (10 points) (a) Identify the redundant vectors in the following list:

$$\begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\-3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

(b) Consider the matrix $A = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$. Compute A^{2016}.

a) $$\begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\1 & 0 & -2 & 1 & 2 & 0\\3 & 0 & -3 & 1 & 4 & 1\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & -2 & 1 & 2 & 0\\-1 & 0 & -1 & 1 & 0 & 0\\3 & 0 & -3 & 1 & 4 & 1\end{bmatrix}\begin{matrix}\\+I\\-3(I)\end{matrix} \rightarrow \begin{bmatrix}1 & 0 & -2 & 1 & 2 & 0\\0 & 0 & -3 & 2 & 2 & 0\\0 & 0 & 3 & -2 & -2 & 1\end{bmatrix}\div -3 \rightarrow$$

6

$$\begin{bmatrix}1 & 0 & -2 & 1 & 2 & 0\\0 & 0 & 1 & -2/3 & -2/3 & 0\\0 & 0 & 3 & -2 & -2 & 1\end{bmatrix}\begin{matrix}+2(II)\\ \\-3(II)\end{matrix} \rightarrow \begin{bmatrix}1 & 0 & 0 & -1/3 & 2/3 & 0\\0 & 0 & 1 & -2/3 & -2/3 & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

correspond to lin. ind. vectors.

$\begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\-2\\-3\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are lin. ind.

so $\boxed{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix} \text{ are redundant}}$

b) $A^2 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$ $A^3 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 4\end{bmatrix} = \begin{bmatrix}1 & 7\\0 & 8\end{bmatrix}$ $A^4 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 7\\0 & 8\end{bmatrix} = \begin{bmatrix}1 & 15\\0 & 16\end{bmatrix}$

4

$$A^n = \begin{bmatrix}1 & 2^n - 1\\0 & 2^n\end{bmatrix}$$

so

$$\boxed{A^{2016} = \begin{bmatrix}1 & 2^{2016}-1\\0 & 2^{2016}\end{bmatrix}}$$