Math 32BH
Winter 2017
Midterm 2
February 27, 2017
Time: 9:00 AM to 9:50 AM

Last Name (Print): Robles

First Name (Print): River

DO NOT OPEN THE EXAM UNTIL INSTRUCTED TO DO SO.

Enter all requested information on the top of this page.

This exam contains 8 pages (including this cover page) and 6 problems.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Manage your time well! If you get stuck on a problem, try working on something else and come back to it later.

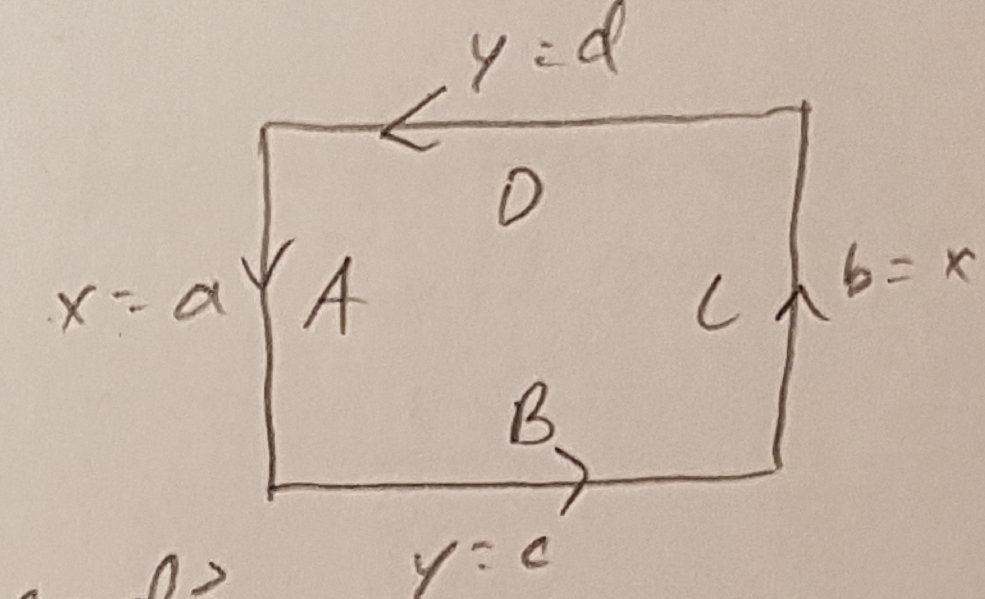
Problem	Points	Score
1	15	15
2	10	10
3	10	10
4	10	6
5	15	15
6	20	20
Total:	80	80

Nice!

1. (15 points) (Partial proof of Green's Theorem) Suppose F is a smooth vector field and $S = [a,b] \times [c,d]$. Prove that

$$\int_{\partial S} Q\hat{j} \cdot d\vec{x} = \int_S \frac{\partial Q}{\partial x}$$

where Q is the y-component of F.



first note: $\int_{\partial S} Q\hat{j} \cdot d\vec{x} = \int_A Q\hat{j} \cdot d\vec{x} + \int_B Q\hat{j} \cdot d\vec{x} + \int_C Q\hat{j} \cdot d\vec{x} + \int_D Q\hat{j} \cdot d\vec{x}$ (see pic.

which we can param. by $\phi_A = \begin{bmatrix} a \\ t \end{bmatrix}$ $t \in [c,d]$

$\phi_B = \begin{bmatrix} t \\ c \end{bmatrix}$ $t \in [a,b]$ $\phi_C = \begin{bmatrix} b \\ t \end{bmatrix}$ $t \in [c,d]$ $\phi_D = \begin{bmatrix} t \\ d \end{bmatrix}$ $t \in [a,b]$

$\Rightarrow \phi_A' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\phi_B' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\phi_C' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\phi_D' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\int_{\partial S} Q\hat{j} \cdot d\vec{x} = \int_d^c \begin{bmatrix} 0 \\ Q(\phi_A) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \int_a^b \begin{bmatrix} 0 \\ Q(\phi_B) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \int_c^d \begin{bmatrix} 0 \\ Q(\phi_C) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \int_b^a \begin{bmatrix} 0 \\ Q(\phi_D) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt$

$= -\int_c^d Q(\phi_A)\,dt + \int_c^d Q(\phi_C)\,dt = \int_c^d Q(b,t) - Q(a,t)\,dt$

also note $\int_S \frac{\partial Q}{\partial x} = \int_c^d \int_a^b \frac{\partial Q}{\partial x}\,dx\,dy$

which by FTOC $= \int_c^d Q(b,y) - Q(a,y)\,dy$

which we see is equivalent to $\int_c^d Q(b,t) - Q(a,t)\,dt$

so we have

$$\int_{\partial S} Q\hat{j} \cdot d\vec{x} = \int_c^d Q(b,t) - Q(a,t)\,dt = \int_S \frac{\partial Q}{\partial x}$$

$\square$.

2. (10 points) Compute $\int_S \vec{F} \cdot \hat{n}$ for $F = \begin{bmatrix} 3y - x \\ 2y + xz \\ -1 \end{bmatrix}$ and S the portion of the unit sphere above the xy-plane oriented up.

Let D be the unit disk oriented down and R ~~the~~ the top half of the unit ball s.t.

$\partial R = S \cup D$

Then $\iint_{\partial R} \vec{F} \cdot \hat{n}\, da = \iint_S \vec{F} \cdot \hat{n}\, da + \iint_D \vec{F} \cdot \hat{n}\, da$

Note: $\operatorname{div}(F) = -1 + 2 + 0 = 1$

and by the Div. Th., $\int_{\partial R} \vec{F} \cdot \hat{n}\, da = \iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R 1\, dV = \operatorname{vol}(R) = \frac{4}{3}\pi \cdot \frac{1}{2}(1)^3$

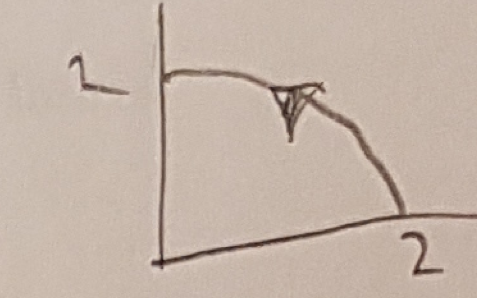
$= \frac{2}{3}\pi$

and $\iint_D \vec{F} \cdot \hat{n}\, da = \iint_D \begin{bmatrix} 3y - x \\ 2y + xz \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} da = \iint_D 1\, da = \pi (1)^2 = \pi$

so $\frac{2}{3}\pi = \iint_S \vec{F} \cdot \hat{n}\, da + \pi$

$\Rightarrow \boxed{\iint_S \vec{F} \cdot \hat{n} = -\frac{1}{3}\pi}$

3. (10 points) Compute $\int_C (x^2 + x + y^2)$ for C given by $x^2 + y^2 = 4$ for $x \geq 0, y \geq 0$.



let $\phi(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix} \Rightarrow \phi' = \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix}$ $t \in [0, \frac{\pi}{2})$

then $\int_c x^2 + x + y^2 = \int_0^{\frac{\pi}{2}} (4\cos^2 t + 2\cos t + 4\sin^2 t) \cdot \sqrt{4\cos^2 t + 4\sin^2 t}\, dt$

$= 2\int_0^{\frac{\pi}{2}} 4 + 2\cos t\, dt = 4 \cdot (2t + \sin t)\Big|_0^{\frac{\pi}{2}}$

$= 4(2\frac{\pi}{2} + \sin(\frac{\pi}{2}) - 2(0) - \sin(0)) = \boxed{4(\pi + 1)}$

$\int_c f\, dt = \int_{t_1}^{t_2} f(\phi)\, |\phi'(t)|\, dt$

4. (10 points) (Multiple choice) Which of the following statements are true for all vector fields, and which are true only for conservative vector fields?

   (a) The line integral around a closed curve is zero.

   A. All vector fields B. Conservative vector fields

   (b) The line integral over a curve only depends on the initial and final points of the curve.

   A. All vector fields B. Conservative vector fields

   (c) The line integral over an oriented curve C does not depend on how C is parametrized (assuming the parametrization preserves the orientation).

   A. All vector fields B. Conservative vector fields

   (d) Reversing orientation of a curve flips the sign of the integral over that curve.

   A. All vector fields B. Conservative vector fields

   (e) The vector field can be written as a gradient of some function.

   A. All vector fields B. Conservative vector fields

5. (15 points) Suppose $A \subset \mathbb{R}$ has zero content. Let $B = \{a + 1 \mid a \in A\}$. Prove that B has zero content.

fix $\varepsilon > 0$, then $\exists$ intervals I_j $j = 1, \dots, n$ $\subseteq \mathbb{R}$ s.t.

$$A \subseteq \bigcup_{j=1}^{n} I_j \text{ and } \sum_{j=1}^{n} m(I_j) < \varepsilon$$

W.r. let $I_j = [a_j, b_j]$

Construct a new set of intervals I_j' with

$$I_j' = [a_j + 1, b_j + 1]$$

Then, if before we had $a_j \leq a \leq b_j \Leftarrow a \in I_j$

$$\Rightarrow a_j + 1 \leq a + 1 \leq b_j + 1$$

$$\Rightarrow a + 1 \in I_j'$$

By induction then, $\forall\, a \in A$, $a \in I_j \Rightarrow a + 1 \in I_j'$

therefore $B \subseteq \bigcup_{j=1}^{n} I_j'$ because all of its elements fall in at least one I_j'

and, $m(I_j) = b_j - a_j$, $m(I_j') = b_j + 1 - (a_j + 1) = b_j - a_j = m(I_j)$, so

$$\sum_{j=1}^{n} m(I_j) < \varepsilon \Rightarrow \sum_{j=1}^{n} m(I_j') < \varepsilon$$

Therefore, B has zero content. $\square$

6. Let $F = \begin{bmatrix} y^2 \\ 2z + x \\ 2y^2 \end{bmatrix}$.

(a) (10 points) Compute **curl** F.

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2z+x & 2y^2 \end{vmatrix} = \hat{i}(4y-2) - \hat{j}(0-0) + \hat{k}(1-2y)$$

$$= \boxed{\begin{bmatrix} 4y-2 \\ 0 \\ 1-2y \end{bmatrix}}$$

(b) (10 points) Find a plane with the equation $ax + by + cz = 0$ (recall that the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is then orthogonal to this plane) such that $\int_C \vec{F} \cdot d\vec{x} = 0$ for all simple closed curves lying on that plane.

if $C = \partial S$

by Stokes Th., $\int_C \vec{F} \cdot d\vec{x} = \iint_S \text{curl}(\vec{F}) \cdot \hat{n} \, da$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \iint_S \begin{bmatrix} 4y-2 \\ 0 \\ 1-2y \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} da = \frac{1}{\sqrt{a^2+b^2+c^2}} \iint_S (4ya - 2a + c - 2yc) \, da$$

which is zero, if $4ya - 2a + c - 2cy = 0$

$\Rightarrow y(4a - 2c) + c - 2a = 0$

and thus $4a - 2c = 0$ and $c - 2a = 0$

which both imply $c = 2a$.

Therefore, we simply need $c = 2a$ regardless of b.

Thus $\boxed{ax + by + 2az = 0 \text{ solves.}}$

for example $x + y + 2z = 0$ solves.