Math 32BH
Winter 2017
Midterm 1
February 6, 2017
Time: 9:00 AM to 9:50 AM

Last Name (Print): Robles

First Name (Print): River

DO NOT OPEN THE EXAM UNTIL INSTRUCTED TO DO SO.

Enter all requested information on the top of this page.

This exam contains 8 pages (including this cover page) and 5 problems.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam.

Do not write in the table to the right.

Manage your time well! If you get stuck on a problem, try working on something else and come back to it later.

Problem	Points	Score	
1	10	10	
2	20	19	-1
3	20	20	
4	20	20	
5	10	9	-1
Total:	80	78	

1. (10 points) Suppose f is a bounded function on a box $B \subset \mathbb{R}^n$. Suppose that for every $\epsilon > 0$ there is a partition P of B such that $S_P f - s_P f < \epsilon$. Prove that f is integrable on B. (You may assume that $\underline{\int}_B f \leq \overline{\int}_B f$.)

10

by the definition of the upper and lower ~~Riemann~~ Riemann Integrals,

$$\overline{\int}_B f = \inf_P \{S_P f\} \quad \underline{\int}_B f = \sup_P \{s_P f\}$$

and thus $\overline{\int}_B f \leq S_P f \;\; \forall P$ ①

and $\underline{\int}_B f \geq s_P f \;\; \forall P$

$\Rightarrow -\underline{\int}_B f \leq -s_P f \;\; \forall P$ ②

adding ① and ② gives us

$$\overline{\int}_B f - \underline{\int}_B f \leq S_P f - s_P f < \varepsilon \text{ as given, so}$$

$$\overline{\int}_B f - \underline{\int}_B f < \varepsilon$$

The given was true $\forall$ arbitrary $\varepsilon > 0$, so letting ε be infinitesimally small gives us

$$\overline{\int}_B f = \underline{\int}_B f$$

which is the def^n for Riemann Integrability. Thus f is integrable. $\square$.

2. Consider function

$$f(x,y) = \begin{cases} 1 & \text{if } x = 0 \\ -1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

1a

(a) (10 points) Let P be a partition of $[0,1] \times [0,2]$ such that $[0,1]$ and $[0,2]$ are both divided into N equal sized subintervals. Compute $S_P f$ and $s_P f$.

note: if $J = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $M_{ij} = 1$ for $i = 1$, j anything $M_{ij} = 0$ otw.

$\Rightarrow S_P f = \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} \left(\frac{1-0}{N}\right)\left(\frac{2-0}{N}\right) = \sum_{j=1}^{N} \frac{1}{N} \cdot \frac{2}{N} = \boxed{\frac{2}{N}}$ ✓

$m_{ij} = -1$ for $i = N$, j anything $m_{ij} = 0$ otw.

$\Rightarrow s_P f = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \left(\frac{1-0}{N}\right)\left(\frac{2-0}{N}\right) = -\frac{2}{N^2} \sum_{j=1}^{N} 1 = \boxed{-\frac{2}{N}}$ ✓

(b) (10 points) Is f integrable on $[0,1] \times [0,2]$? Prove your answer.

f is integrable if $\forall \varepsilon > 0 \ \exists P$ s.t. $S_P f - s_P f < \varepsilon$] not the def'n, but ok. (Prob #1)

$S_P f - s_P f = \frac{2}{N} - \left(-\frac{2}{N}\right) = \frac{4}{N}$

for any $\varepsilon > 0$, choose P s.t. $N = \frac{8}{\varepsilon}$ $\in \mathbb{Z}$?

−1

then $S_P f - s_P f = \frac{4}{8}\varepsilon = \frac{1}{2}\varepsilon < \varepsilon$

$\Rightarrow f$ is integrable on $B = [0,1] \times [0,2]$ $\square$.

3. (a) (10 points) Suppose we have $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, $I \subset \mathbb{R}$, and $c \in \mathbb{R}$. Let $A = \{f(p) + c \mid p \in I\}$ and $B = \{f(p) \mid p \in I\}$. Prove that $\sup A \leq c + \sup B$ (you may assume that both supremums exist).

by def.

$\Rightarrow \forall\, p \in I,$

$f(p) \leq \sup B$

$\Rightarrow f(p) + c \leq \sup B + c$

therefore $\sup B + c$ is an upper bound for $A = \{f(p) + c \mid p \in I\}$

but $\sup A$ is defined as the least upper bound, so we must have

$\sup A \leq \sup B + c$ $\square$

(b) (10 points) Suppose f is integrable (and bounded) on $[a,b]$, P partitions $[a,b]$, and $c \in \mathbb{R}$. Prove that $S_P(f+c) \le c(b-a) + S_P f$.

$$S_P f = \sum_{i=1}^{n} M_i \Delta x_i \text{ for } P = \{x_0, x_1, \dots, x_n\} \quad x_0 = a,\ x_n = b$$

where $M_i := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

$$S_P(f+c) = \sum_{i=1}^{n} N_i (f+c) \Delta x_i$$

where $N_i := \sup\{f(x)+c \mid x \in [x_{i-1}, x_i]\}$.

but by part a, $N_i \le M_i + c \quad \forall\, i$, so we have

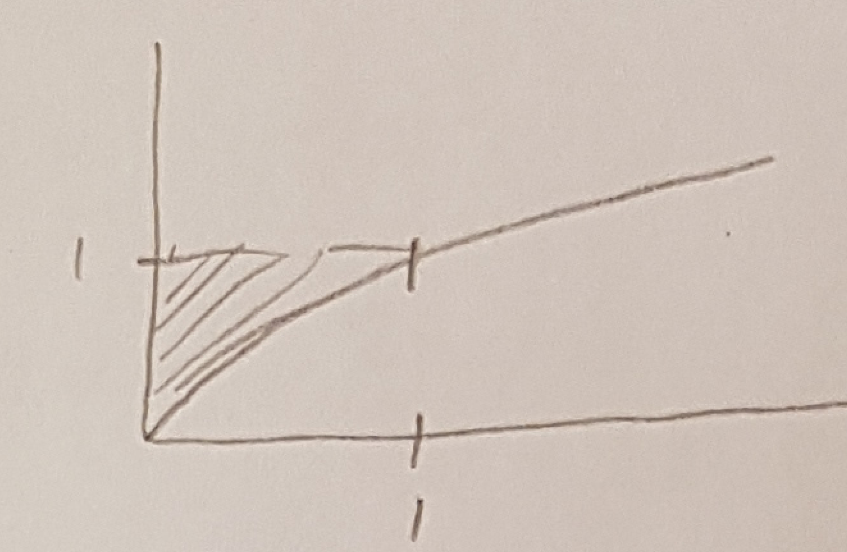
$$S_P(f+c) = \sum_{i=1}^{n} N_i \Delta x_i \le \sum_{i=1}^{n} (M_i + c)\Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i + \sum_{i=1}^{n} c \Delta x_i$$

$$= S_P f + c \sum_{i=1}^{n} \Delta x_i = S_P f + c(b-a)$$

thus $S_P(f+c) \le S_P f + c(b-a)$ $\square$.

4. (a) (10 points) Compute

$$\int_0^1 \int_{\sqrt{x}}^1 3\sin(y^3-1)\, dy dx$$



let y run from 0 to 1
x from 0 to y^2

$\Rightarrow \int_0^1 \int_0^{y^2} 3\sin(y^3-1)\, dx dy = \int_0^1 3(x|_0^{y^2})\sin(y^3-1)\, dy = \int_0^1 3y^2 \sin(y^3-1)\, dy$

if $u = y^3 - 1$
$du = 3y^2 dy$ $\Rightarrow \int \sin u\, du = -\cos(u)| = -\cos(y^3-1)|_0^1$

$= -\cos(0) + \cos(-1) = \boxed{\cos(-1) - 1}$

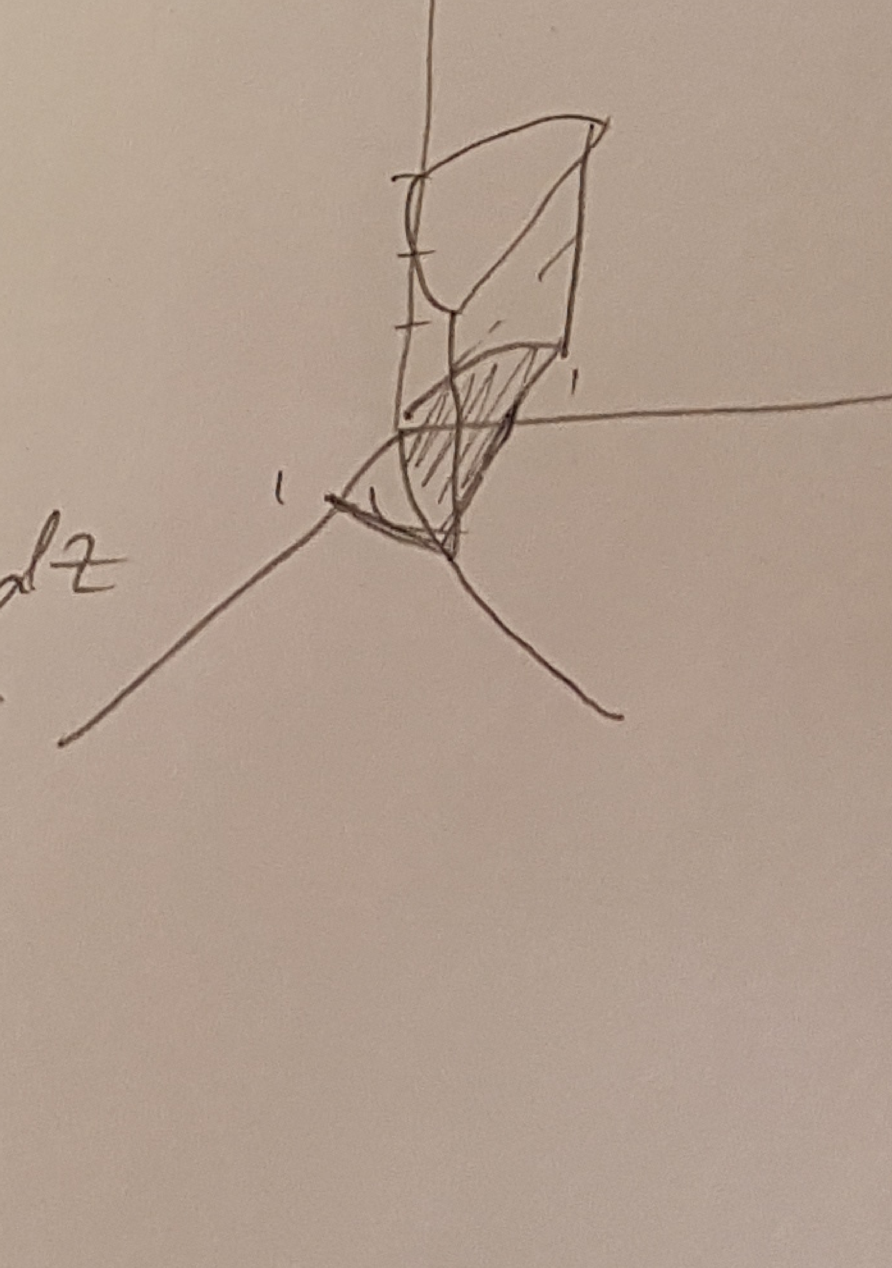
(b) (10 points) Compute the volume of the solid bounded by $y = x^2$, $z = 0, z = 3, y = 1$.

$x: [-1,1]$ $y: [x^2, 1]$ $z: [0,3]$

$\int_0^3 \int_{-1}^1 \int_{x^2}^1 dy\, dx\, dz = \int_0^3 \int_{-1}^1 (1-x^2)\, dx\, dz$

$= \int_0^3 (x - \frac{1}{3}x^3)|_{-1}^1\, dz = \int_0^3 (1 - \frac{1}{3} - (-1) + (-\frac{1}{3}))\, dz$

$= 3(2 - \frac{2}{3}) = 6 - 2 = \boxed{4}$



5. (10 points) Let $\Phi(u,v) = (u^2 - v^2, 2uv)$. Compute

$$\int_{\Phi([0,1]^2)} y$$

$$J_\Phi = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$$

$$= \int_{[0,1]^2} 2uv \left| \det J_\Phi(u,v) \right| du\,dv$$

$$\Rightarrow \det J_\Phi = 4u^2 + 4v^2 = 4(u^2+v^2)$$

$$\Rightarrow \left| \det J_\Phi \right| = 4\left|u^2+v^2\right| = 4(u^2+v^2)$$

$$= \int_0^1 \int_0^1 2uv \cdot 4(u^2+v^2)\, du\,dv$$

$$= 8 \int_0^1 \int_0^1 u^3 v + uv^3\, du\,dv$$

$$= 8 \int_0^1 \frac{1}{4}u^4 v + \frac{1}{2}u^2 v^3 \Big|_0^1 dv = \overset{2}{\cancel{4}} \int_0^1 v + 2v^3\, dv$$

$$= 4\left(\frac{1}{2}v^2 + \frac{2}{4}v^4\right)\Big|_0^1 = 4\left(\frac{1}{2} + \frac{1}{2}\right) = \boxed{4}$$

-1