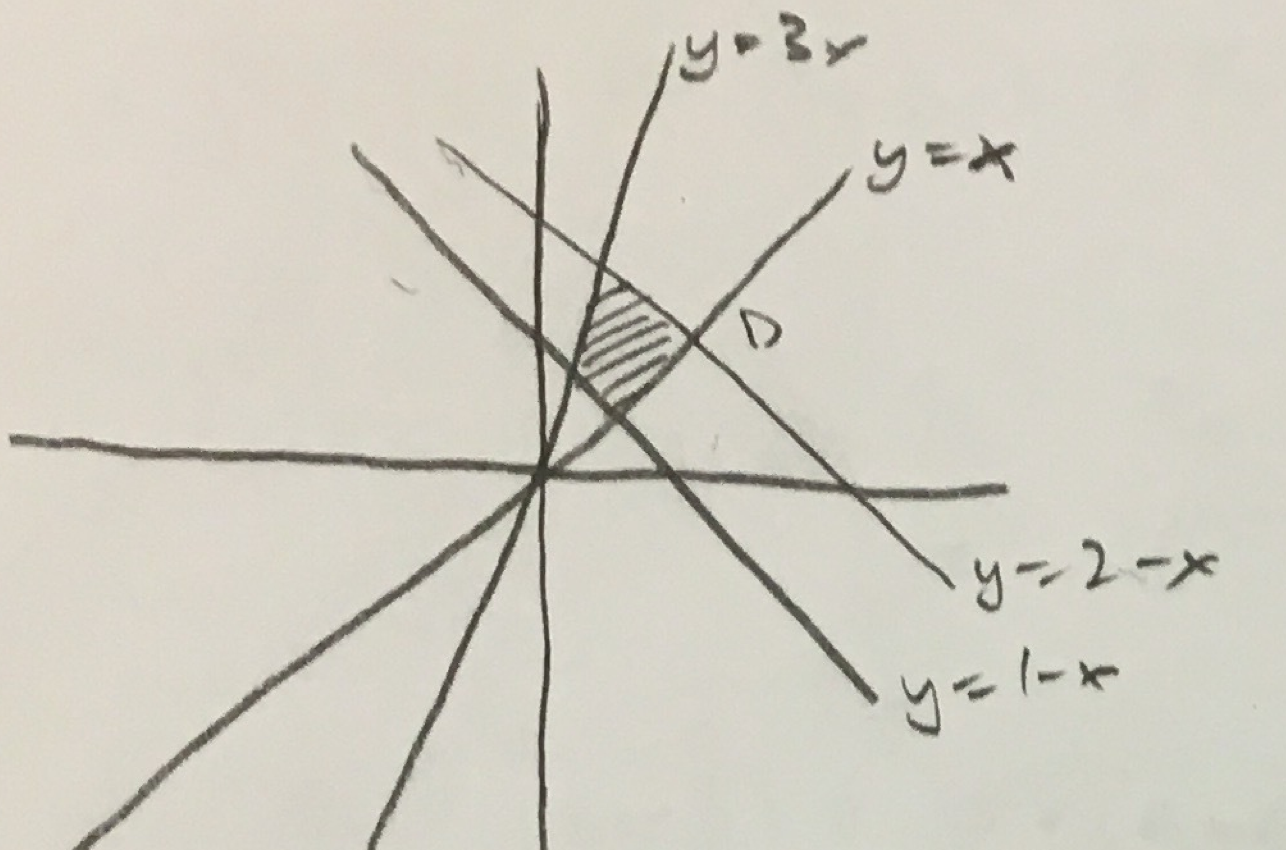
Problem 1. (15 points) Let D be the region enclosed by $y = x$, $y = 3x$, $y = 1 - x$, $y = 2 - x$.
(a) (5 points) Find a map $F(x, y)$ whose image $F(D)$ is a rectangle (i.e., maps D to a rectangle)
(b) (10 points) Evaluate $\iint_D \frac{y(x+y)}{x^3}\, dxdy$ using change of variables from F.

~~AB=CD~~

$y = x \quad y = 3x \quad y = 1 - x \quad y = 2 - x$

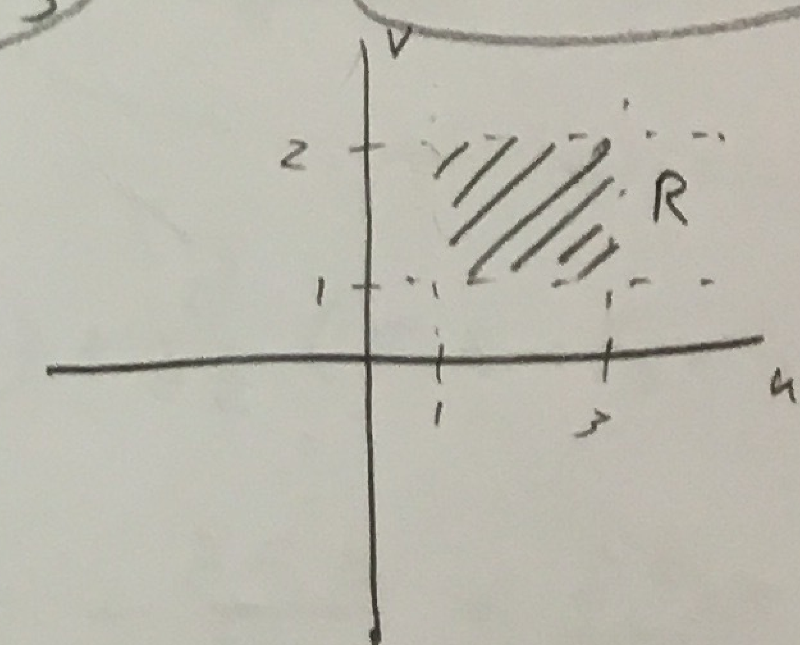


$3x > y > x \qquad 2 - x > y > 1 - x$

$3 > \frac{y}{x} > 1 \qquad 2 > y + x > 1$

$1 \le \frac{y}{x} \le 3 \qquad 1 \le y + x \le 2$

$1 \le u \le 3 \qquad 1 \le v \le 2$



a) $\boxed{\begin{array}{l} F(x, y) = \left(\overset{u}{\frac{y}{x}}, \overset{v}{y + x}\right) \\ u(x,y) = y/x \quad v(x,y) = y + x \end{array}}$

b) $|Jac(G)(u,v)| = \dfrac{1}{|Jac(F)(x,y)|}$

$Jac(F)(x,y) = \det \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \det \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 1 \end{vmatrix} = \left| -\frac{y}{x^2} - \frac{1}{x} \right| = \frac{y}{x^2} + \frac{1}{x} = \frac{y + x}{x^2}$

$|Jac(G)(u,v)| = \dfrac{1}{\frac{y+x}{x^2}} = \dfrac{x^2}{y + x} = Jac(G)$

$\iint \frac{y(x+y)}{x^3} |Jac(G)(u,v)|\, dudv = \iint_R \left(\frac{y(x+y)}{x^3}\right)\left(\frac{x^2}{(y+x)}\right) dudv = \iint_R \frac{y}{x}\, dudv = \iint_R u\, dudv$

$R: 1 \le u \le 3 \quad 1 \le v \le 2$

$= \int_1^2 \int_1^3 u\, dudv = \frac{1}{2} \int_1^2 u^2 \Big|_1^3 dv = \frac{1}{2} \int_1^2 8\, dv = \frac{1}{2}(8)(2-1) = \boxed{4}$ units²

Problem 2. (15 points) Let a, b, c be real constants. Show that $\operatorname{div}(\mathbf{F} \times \langle a, b, c\rangle) = 0$ if $\mathbf{F}$ is a conservative vector field.

a, b, c constant

↓

no curl × <a, b, c> → const

div(const) = 0

~~div(curl(F)) = 0~~

~~curl(div(F)) = 0~~

$$F = \langle \overset{P}{F_1} \;\; \overset{Q}{F_2} \;\; \overset{R}{F_3} \rangle = \langle P, Q, R \rangle$$

$$\to F \times \langle a, b, c \rangle = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ a & b & c \end{vmatrix} = (F_2 c - F_3 b)i - (F_1 c - F_3 a)j + (F_1 b - F_2 a)k$$

$P_y = Q_x \quad P_z = R_x \quad Q_z = R_y$

$$\to \operatorname{div}(F \times \langle a, b, c\rangle) = \nabla \cdot (F \times \langle a, b, c \rangle)$$

$$= \frac{\partial}{\partial x}(F_2 c - F_3 b) + \frac{\partial}{\partial y}(F_3 a - F_1 c) + \frac{\partial}{\partial z}(F_1 b - F_2 a)$$

$$= \frac{\partial F_2 c}{\partial x} - \frac{\partial F_3 b}{\partial x} + \frac{\partial F_3 a}{\partial y} - \frac{\partial F_1 c}{\partial y} + \frac{\partial F_1 b}{\partial z} - \frac{\partial F_2 a}{\partial z}$$

$$= c\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) + b\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + a\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)$$

$$\to F = \langle F_1, F_2, F_3 \rangle = \langle P, Q, R \rangle$$

$$\to = c(Q_x - P_y) + b(P_z - R_x) + a(R_y - Q_z)$$

If F is conservative: $Q_x = P_y$
$P_z = R_x$
$R_y = Q_z$

So, $\to c(0) + b(0) + a(0) = \boxed{0}$

$$\therefore \boxed{\operatorname{div}(F \times \langle a, b, c \rangle) = 0}$$

Problem 3. (20 points) Consider the path C parametrized by $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ for $0 \le t \le 1$.
(a) (10 points) Evaluate the length of C.

(b) (10 points) Evaluate the vector line integral $\int_C \mathbf{F}\, d\mathbf{r}$, where $\mathbf{F} = \langle -y, x, z \rangle$.

$r(t) = (\cos 2t, \sin 2t, t)$

a) $\text{Length} = \int_0^1 \| r'(t) \|\, dt = \int_0^1 \| \langle -2\sin 2t, 2\cos 2t, 1 \rangle \|\, dt$

$\sqrt{4\sin^2 2t + 4\cos^2 2t + 1} = \sqrt{5}$

$= \int_0^1 \sqrt{5}\, dt = \boxed{\sqrt{5}}$

b) $\int_C F\, dr = \int_C \langle -y, x, z \rangle \cdot r'(t)\, dt = \int_0^1 \langle -\sin 2t, \cos 2t, t \rangle \cdot \langle -2\sin 2t, 2\cos 2t, 1 \rangle\, dt$

$= \int_0^1 (2\sin^2 2t + 2\cos^2 2t + t)\, dt = \int_0^1 (t+2)\, dt = \cancel{}$

$= \left(\frac{t^2}{2} + 2t\right)\Big|_0^1 = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$

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checkwork

$F = \langle -y, x, z \rangle = \langle -\sin 2t, \cos 2t, t \rangle \cdot \langle -2\sin 2t, 2\cos 2t, 1 \rangle$

$2\sin^2 2t + 2\cos^2 2t + t$

$\int_0^1 (2 + t)\, dt$

$\frac{t^2}{2} + 2t \Big|_0^1 = \frac{1}{2} + 2 = \frac{4}{2} + \frac{1}{2} = 5/2$

Problem 4. (20 points) Let C be a path from $(2,0)$ to $(0,1)$ along the ellipse $x^2+4y^2=4$ in the first quadrant, oriented counterclockwise.

(a) (10 points) Let $\mathbf{F} = \langle -y\sin x \overset{+y}{,} x + \cos x \rangle$. Show that $\mathbf{F}$ is conservative, find a potential function $f(x,y)$ so that $\mathbf{F} = \nabla f$, and evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

(b) (10 points) Let $\mathbf{F} = \langle -y, x\rangle$. Is it conservative? Evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

a) $\vec{F} = \langle \underset{P}{y - y\sin(x)},\ \underset{Q}{x + \cos(x)} \rangle$

$P_y = Q_x$

$1 - \sin(x) = 1 - \sin(x)$ ✓

Because $P_y = Q_x$ and the domain of F is simply connected (there are no holes or abnormalities anywhere) then F must be conservative.

$\nabla f(x,y) = F$

$\boxed{f(x,y) = y\cos x + xy}$

$f_x = y - y\sin x$ ✓

$f_y = x + \cos x$ ✓

Because conservative,....

$\int_C F\,dr = f(B) - f(A) = f(0,1) - f(2,0)$

$= [1\cos 0 + (0)(1)] - [0\cos 2 + (2)(0)]$

$\boxed{\int_C F dr = 1}$

b) $F = \langle -y, x\rangle$

$\text{curl}(\vec{F}) \neq 0$

$\boxed{\therefore F = \langle -y, x\rangle \text{ Is not conservative}}$

$f(x,y)$ D.N.E

pt. fcn $\exists$

$\int_C F\,dr$: param-n $C = \langle 2\cos t, \sin t\rangle$ $0 \le t \le \pi/2$

$\int_C F dr = \int_0^{\pi/2} \langle -\sin t, 2\cos t\rangle \cdot \langle -2\sin t, \cos t\rangle\, dt = \int_0^{\pi/2} (2\sin^2 t + 2\cos^2 t)\, dt$

$= \int_0^{\pi/2} 2\, dt = \boxed{\pi}$

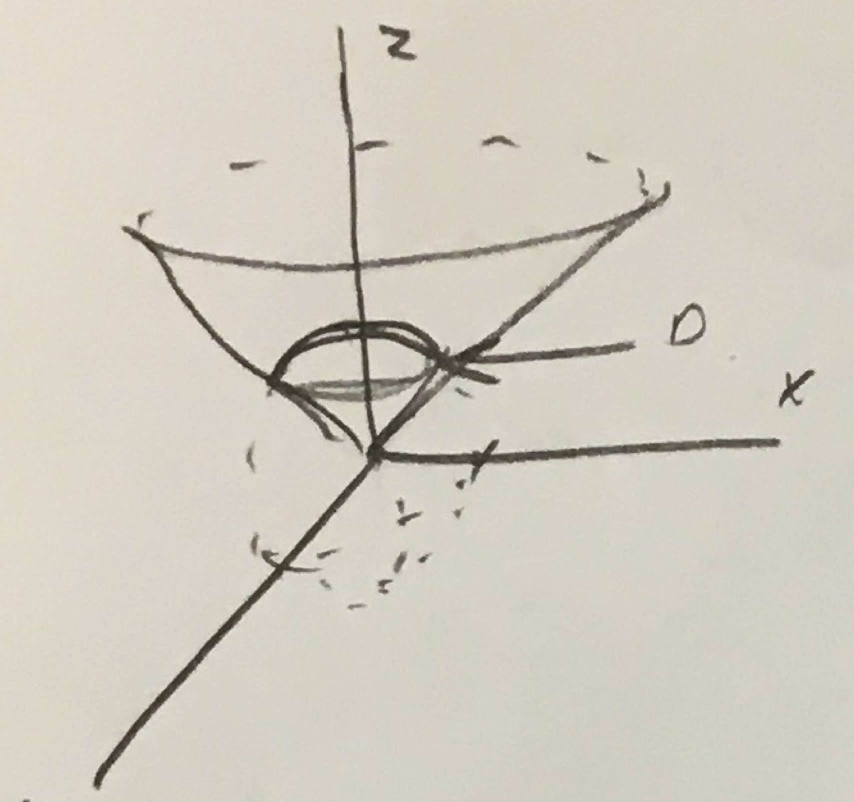
Problem 5. (20 points) Compute the area of the surface enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$ from above.

$z = \sqrt{x^2+y^2} \rightarrow z^2 = x^2+y^2$

$x^2+y^2+z^2 = 1$

Sum of two surface integrals

$\hookrightarrow \iint_{sphere} + \iint_{cone}$

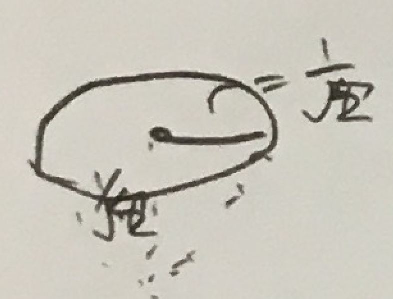


$x^2+y^2+x^2+y^2 = 1$

$x^2+y^2 = \frac{1}{2}$

$r = \frac{1}{\sqrt{2}}$

Top of sphere



$G(\theta, \varphi) = (R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi)$

$= (\frac{1}{\sqrt{2}}\cos\theta\sin\varphi, \frac{1}{\sqrt{2}}\sin\theta\sin\varphi, \frac{1}{\sqrt{2}}\cos\varphi)$

$\tan^{-1}(1) = \pi/4$

$\|\vec{N}(\theta,\varphi)\| = \|G_\theta \times G_\varphi\| = R^2\sin\varphi = \frac{\sin\varphi}{2}$ NO

derive

$\int_0^{2\pi}\int_0^{\pi/4} 1(\|\vec{N}(\theta,\varphi)\|) = \int_0^{2\pi}\int_0^{\pi/4} \frac{\sin\varphi}{2}\, d\varphi\, d\theta = \frac{1}{2}\int_0^{2\pi} -\cos\theta \Big|_0^{\pi/4} d\theta$

$= \frac{1}{2}\int_0^{2\pi} 1 - \frac{\sqrt{2}}{2}\, d\theta = \pi\left(1 - \frac{\sqrt{2}}{2}\right)$

Cone S.A.

$G(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$\|\vec{N}(r,\theta)\| = \|G_r \times G_\theta\| = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \|\langle -r\cos\theta, -r\sin\theta, r\rangle\| = \underbrace{\sqrt{r^2 + r^2}}\ = \sqrt{2}\,r$

$\int_0^{2\pi}\int_0^{1/\sqrt{2}} 1(\sqrt{2}\,r)\, dr\, d\theta = 2\pi\left(\frac{\sqrt{2}}{2} r^2 \Big|_0^{1/\sqrt{2}}\right) = \pi\sqrt{2}\left(\frac{1}{2} - 0\right) = \frac{\pi\sqrt{2}}{2}$ ✓

Total S.A. $= \pi - \frac{\pi\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{2} = \boxed{\pi}$ units^2