Midterm 1 Dr. Coroniti

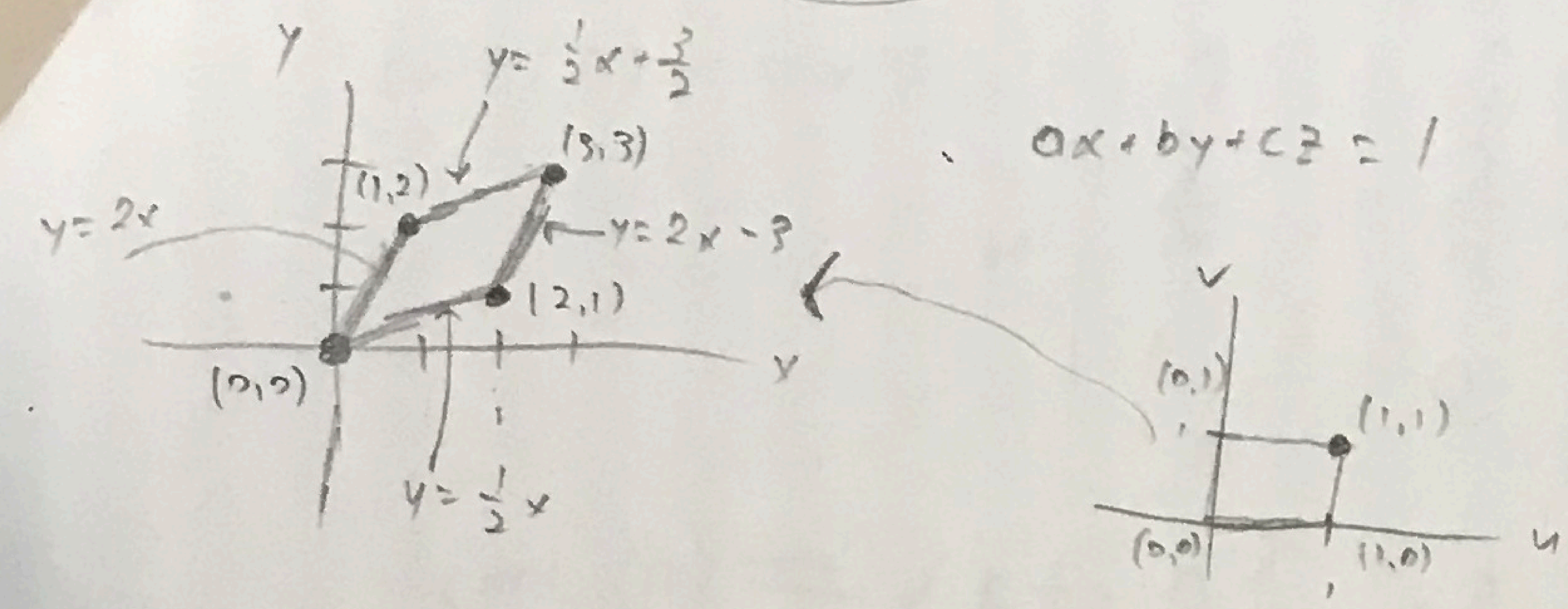
s on the exam, and you have 50 minutes. To receive full credit,
... will be given for answers that simply

32B-1 Yeliussizov. Final exam

...ne: 8:00 AM - 11:00 AM, June 10, 2017

**Problem 1.** (10 points) Let $D$ be the parallelogram in the $xy$-plane with vertices $(0,0),(2,1),(1,2),(3,3)$.
(a) (5 points) Find a linear map $G(u,v)$ that maps the unit square $[0,1]\times[0,1]$ in the $uv$-plane to $D$ in the $xy$-plane.

(b) (5 points) Evaluate $\iint_D (x+y)\,dxdy$ using change of variables given by $G(u,v)$.




$ax+by+cz=1$

$0\le u\le 1$

$0\le v\le 1$

$2x \le y \le 2x-3$

$\frac{1}{2}x \le y \le \frac{1}{2}x+\frac{3}{2}$

$x \le 2y \le x+3$

$0\le 2y-x\le 3$

$0 \le y-2x \le -3$

$-3 \le 2x-y \le 0$

$\text{Jac}(G) = \begin{matrix} X_u & X_v \\ Y_u & Y_v \end{matrix}$

$G(u,v) = (2u+v,\ 2v+u)$ ✓

| u v | x y |
|---|---|
| (0,0) | (0,0) ✓ |
| (1,0) | (2,1) ✓ |
| (1,1) | (3,3) ✓ |
| (0,1) | (1,2) ✓ |

$$\int_{u=0}^{1}\int_{v=0}^{1} (2u+v)+(2v+u)\ \text{Jac}(G)\,du\,dv$$

$$\text{Jac}(G)(u,v) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3$$

$$3\int_{u=0}^{1}\int_{v=0}^{1} 3u+3v\,dv\,du = 9\int_{u=0}^{1}\int_{v=0}^{1}(u+v)\,dv\,du$$

$$9\int_{u=0}^{1}\left(uv+\frac{v^2}{2}\Big|_{v=0}^{1}\right)du$$

$$9\int_{u=0}^{1} u+\frac{1}{2}\,du = 9\left(\frac{u^2}{2}+\frac{1}{2}u\Big|_{u=0}^{1}\right) = 9\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$=9(1)$$

$$=9$$

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**Problem 2.** (10 points) Let $\mathbf{F} = \langle P, Q, R\rangle$ be an arbitrary vector field (where $P, Q, R$ are continuous differentiable functions). Show that $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$.

$$\operatorname{curl}(F) = \nabla \times F \overset{\text{def}}{=} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\operatorname{curl}(F) = \underbrace{\left\langle R\frac{\partial}{\partial y} - Q\frac{\partial}{\partial z},\ P\frac{\partial}{\partial z} - R\frac{\partial}{\partial x},\ Q\frac{\partial}{\partial x} - P\frac{\partial}{\partial y} \right\rangle}$$

$$\operatorname{div}(\operatorname{curl}(F)) = \nabla \cdot \langle \quad \rangle$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left\langle R\frac{\partial}{\partial y} - Q\frac{\partial}{\partial z},\ P\frac{\partial}{\partial z} - R\frac{\partial}{\partial x},\ Q\frac{\partial}{\partial x} - P\frac{\partial}{\partial y} \right\rangle$$

$$= R\cancel{\frac{\partial^2}{\partial x \partial y}} - Q\cancel{\frac{\partial^2}{\partial x \partial z}} + P\cancel{\frac{\partial^2}{\partial y \partial z}} - R\cancel{\frac{\partial^2}{\partial y \partial x}} + Q\cancel{\frac{\partial^2}{\partial x \partial z}} - P\frac{\partial^2}{\partial y \partial z}$$

$$= 0$$

which terms get cancelled ?

9

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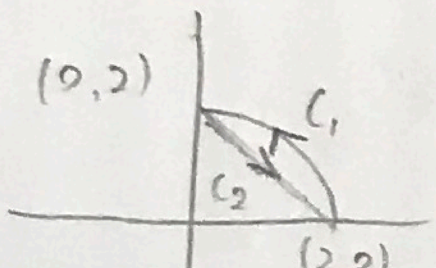
**Problem 3.** (15 points) Let $C$ be the closed path from $(2,0)$ to $(0,2)$ along the quarter circle $x^2+y^2=4$ (counterclockwise), and then going back from $(0,2)$ to $(2,0)$ along the straight line segment.

(a) (7 points) Evaluate the scalar integral $\int_C y\, ds$ $= \int_{C_1} y\,ds + \int_{C_2} y\,ds$

(b) (8 points) Evaluate the vector line integral $\int_C \langle 2y, x\rangle\, d\mathbf{r}$




$C_1$ $r_1(t) = \langle 2\cos t, 2\sin t\rangle$ $0 \le t \le \frac{\pi}{2}$ $r_1'(t) = \langle -2\sin t, 2\cos t\rangle$ $\|r_1'(t)\| = \sqrt{4} = 2$

$C_2$ $r_2(t) = \langle t, 2-t\rangle$ $0 \le t \le 2$ 2 parts

$y = 2-x$ $t = 2-x$

$y = 2-t$ $x = 2-t$

$r_2'(t) = \langle 1, -1\rangle$ $\|r_2'(t)\| = \sqrt{1^2+1^2} = \sqrt{2}$

(a) $\int_C y\,ds = \int y(r_1(t))\cdot\|r_1'(t)\|\,dt + \int y(r_2(t))\cdot\|r_2'(t)\|\,dt$

$= 2\int_{t=0}^{\frac{\pi}{2}} 2\sin t\,dt + \sqrt{2}\int_{t=0}^{2}(2-t)\,dt$

$4\left(-\cos t\Big|_{t=0}^{\frac{\pi}{2}}\right) + \sqrt{2}\left(2t - \frac{t^2}{2}\Big|_{t=0}^{2}\right)$

$4(0+1) + \sqrt{2}(4-2) = 4 + 2\sqrt{2}$

$\int_C \langle 2y, x\rangle\,dr = \int_{C_1}\langle 2y, x\rangle\,dr + \int_{C_2}\langle 2y, x\rangle\,dr$ $\int_D F(r(t))\cdot r'(t)\,dt$

$\int_{t=0}^{\frac{\pi}{2}}\langle 4\sin t, 2\cos t\rangle\cdot\langle -2\sin t, 2\cos t\rangle\,dt + \int_{t=0}^{2}\langle 4-2t, t\rangle\cdot\langle 1,-1\rangle\,dt$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{2i\theta} = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$\int_{t=0}^{\frac{\pi}{2}}(-8\sin^2 t + 4\cos^2 t)\,dt + \int_{t=0}^{2} 4-2t-t\,dt$

$\sin^2 = 1-\cos^2$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$4\int_0^{\frac{\pi}{2}} -2(1-\cos^2) + \cos^2 t\,dt + \int_{t=0}^{2} 4-3t\,dt$

$\frac{12}{2}\int \cos 2\theta + 1\,dt$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$4\int_0^{\frac{\pi}{2}} 3\cos^2 t - 2\,dt$

$6\int_0^{\frac{\pi}{2}}\cos 2\theta + 6\int_0^{\frac{\pi}{2}} 1\,dt$

$\cos 2\theta = 1 - 2\sin^2\theta$

$+ 6\left(\frac{\pi}{2}\right) = 3\pi$

$4t - \frac{3t^2}{2}\Big|_{t=0}^{2}$

$\frac{\cos 2\theta - 1}{-2} = \sin^2\theta$

$12\int_0^{\frac{\pi}{2}}\cos^2 t - 8\int_0^{\frac{\pi}{2}} dt$ $+$

$u = 2\theta$ $u(\frac{\pi}{2}) = \pi$ $u(0) = 0$ $du = 2d\theta$ $\frac{1}{2}du = d\theta$

$+$

$+ (8-6) =$

$3\int_0^{\pi}\cos u\,du$

$\frac{1 - \cos 2\theta}{2} = \sin^2\theta$

$-8\left(\frac{\pi}{2}\right)$

$2$

$3\sin u\Big|_{u=0}^{\pi} + 3\pi$

$3 + 3\pi - 4\pi + 2 = 5 - \pi$

$3 + 3\pi$

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roblem 4. (10 points) Consider the vector field $\mathbf{F} = \langle \sin y, x\cos y + \cos z, -y\sin z + \cos z\rangle$.
Evaluate $\int_C \mathbf{F}\, d\mathbf{r}$, where $C$ is the path given by $\mathbf{r}(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi/2$.

$f = (x\sin y + y\cos z + \sin z)$

is it conservative? $\nabla f = F$

$\frac{\partial f}{\partial x} = \sin y$

$\frac{\partial f}{\partial y} = x\cos y + \cos z$

$\frac{\partial f}{\partial z} = -y\sin z + \cos z$

So $\int_C F\,dr = f(r(\frac{\pi}{2})) - f(r(0)) = 1 - 0 = 1$

$r(\frac{\pi}{2}) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2}, \frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2}\rangle$

$r(0) = (\cos(0), \sin(0), 0) = \langle 1, 0, 0\rangle$

$f(0, 1, \frac{\pi}{2}) = 1$

$f(1, 0, 0) = 0$

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**Problem 5.** (15 points) Let $S$ be the part of the plane $z = 2x$ contained in the paraboloid $z = x^2 + y^2$.

(a) (8 points) Evaluate the area of $S$. 4

(b) (7 points) Evaluate $\iint_S x \, dS$ 0

$z = 2r\cos\theta$ $\quad$ $z = r^2$

(a) $\iint_D \|G_u \times G_v\| \, du\, dv$

on $x$-$z$ plane

$0 \leq x \leq 2$

$x^2 \leq z \leq 2x$

$x^2 + y^2 \leq z \leq 2x$

$r^2 \leq z \leq 2r\cos\theta$

$0 \leq \frac{z}{2} \leq 2\cos\theta$

Area $(S) = \iint_S dS = \iint_D \|G_x \times G_y\|$

$z = 2x$

$z = x^2 + y^2$

$2x = x^2 + y^2$

$x^2 + y^2 - 2x = 0$

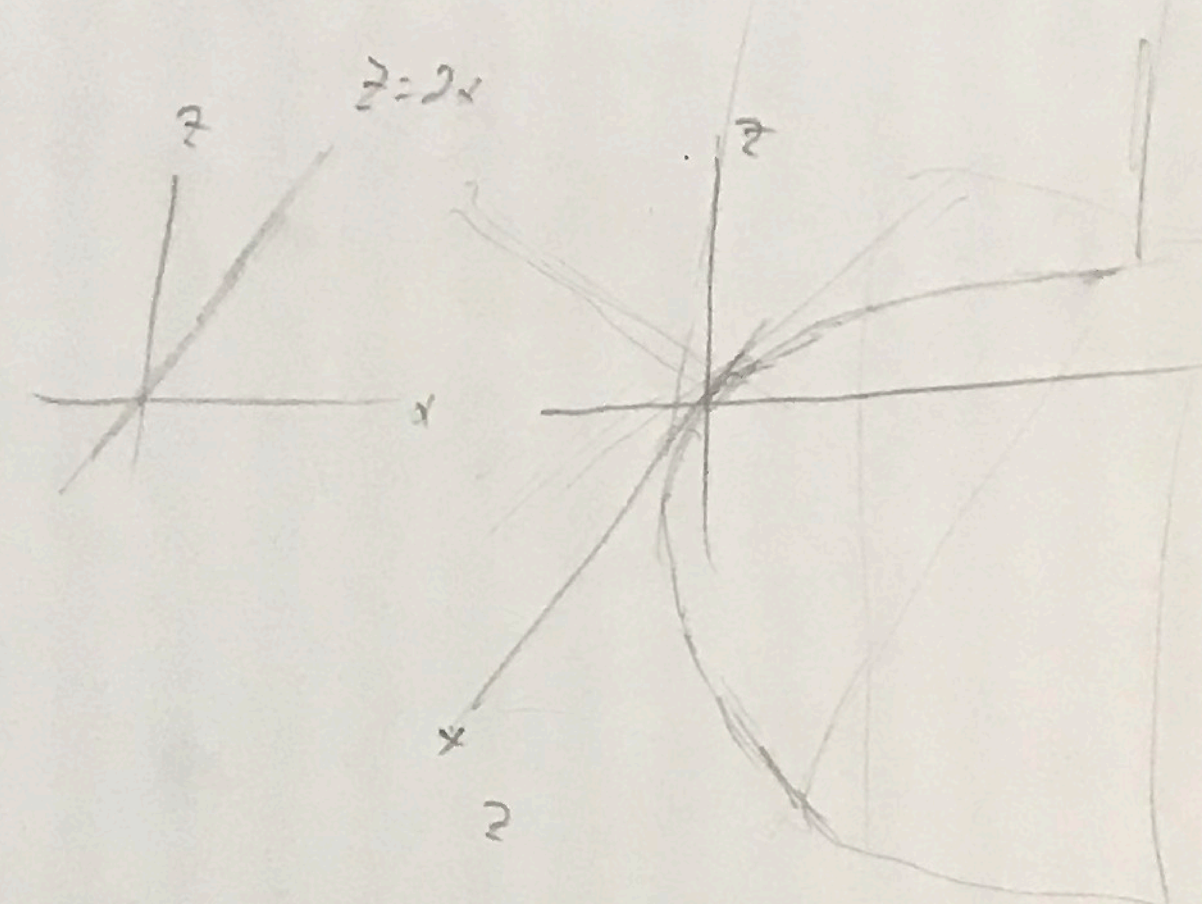
For paraboloid

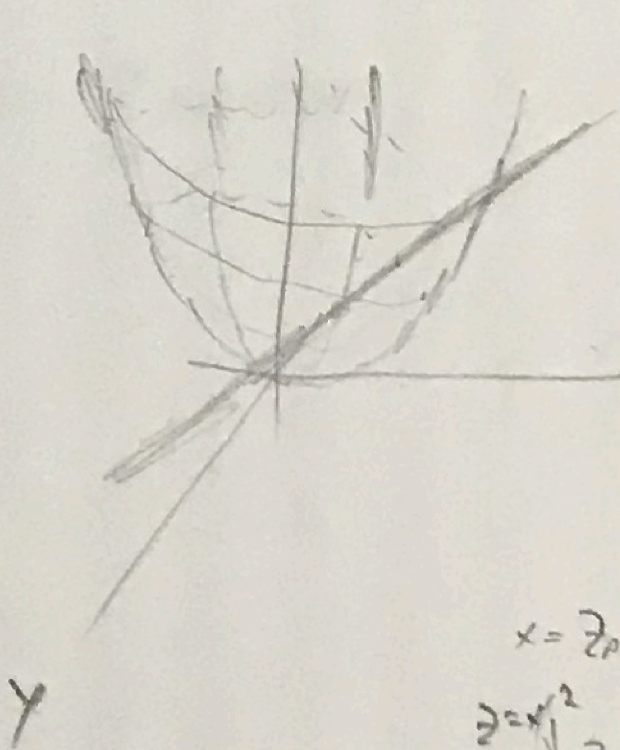
$G(x,y) = (x, y, x^2 + y^2)$ +2

$\{G_x \times G_y\} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle$

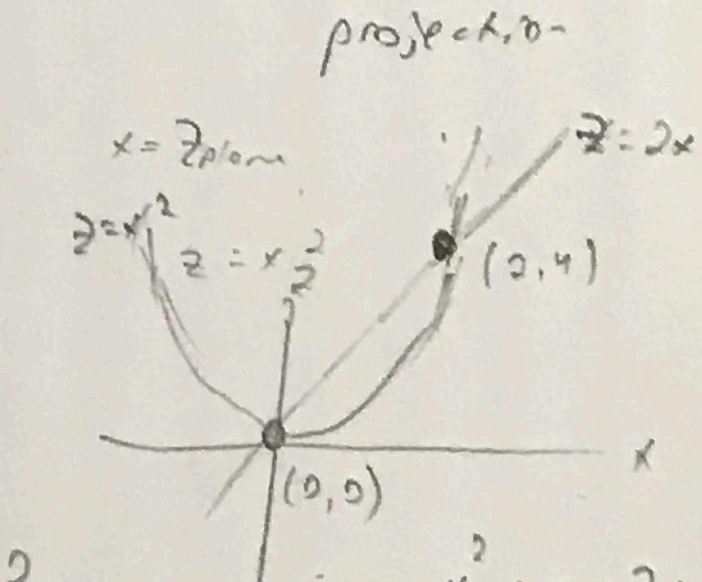
$\|G_x \times G_y\| = \sqrt{4x^2 + 4y^2 + 1}$ +2

$= \sqrt{4r^2 + 1}$




projection



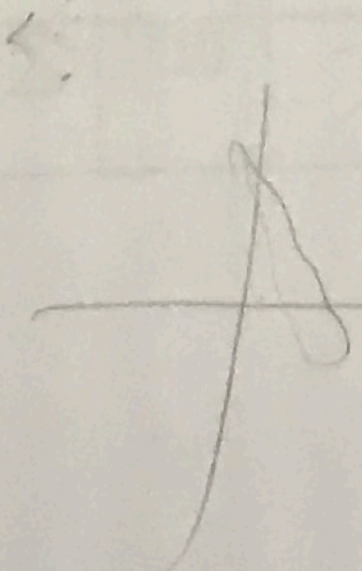

$\frac{x^2}{x} = \frac{2x}{x}$ $\quad$ $x = 2, 0$

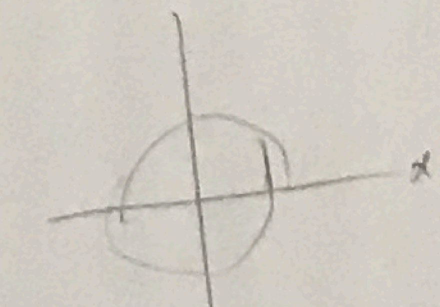
$x^2 \leq z \leq 2x$

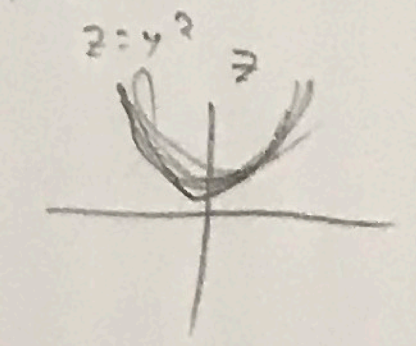
$0 \leq x \leq 2$

y-z plane

$z = y^2$







$r\,dr$ 2

(b) $\iint_S x \, dS = \iint_D G(x,y)(x) \|G_x \times G_y\| \, dx\, dy$