Problem 1. (15 points) Evaluate the double integral over the given rectangular domain in the xy-plane

$$\iint_R (1 + y + xe^{xy})dA, \qquad R = [0,2] \times [-1,1].$$

$$\int_0^2 \int_{-1}^1 (1 + y + xe^{xy})\, dy\, dx$$

$$\int_0^2 \left(y + \frac{y^2}{2} + e^{xy}\right)\Big|_{-1}^{1} dx$$

$$\int_0^2 \left(1 + \frac{1}{2} + e^x\right) - \left(-1 + \frac{1}{2} + e^{-x}\right) dx$$

$$\int_0^2 \left(1 + \frac{1}{2} + e^x + 1 - \frac{1}{2} - e^{-x}\right) dx$$

$$\int_0^2 2 + e^x - e^{-x}\, dx$$

$$\left(2x + e^x + e^{-x}\right)\Big|_0^2 = 4 + e^2 + e^{-2} - 1 - 1$$

$$\boxed{(2 + e^2 + e^{-2}) \text{ units}^3}$$

✓

Problem 2. (15 points) Let D be the region bounded by $y = 2x$ and $y = x^2$.
(a) (5 points) Sketch the region D in the xy-plane.
(b) (10 points) Compute the double integral of $f(x, y) = \dfrac{x}{4-y}$ over the domain D. (Choose the order of integration that enables you to evaluate the integral.)

$y = 2x \qquad y = x^2$

either $\rightarrow \displaystyle\int_0^2\int_{x^2}^{2x} \frac{x}{4-y}\,dy\,dx$

$x\ln(4-y)\Big|_{x^2}^{2x}$

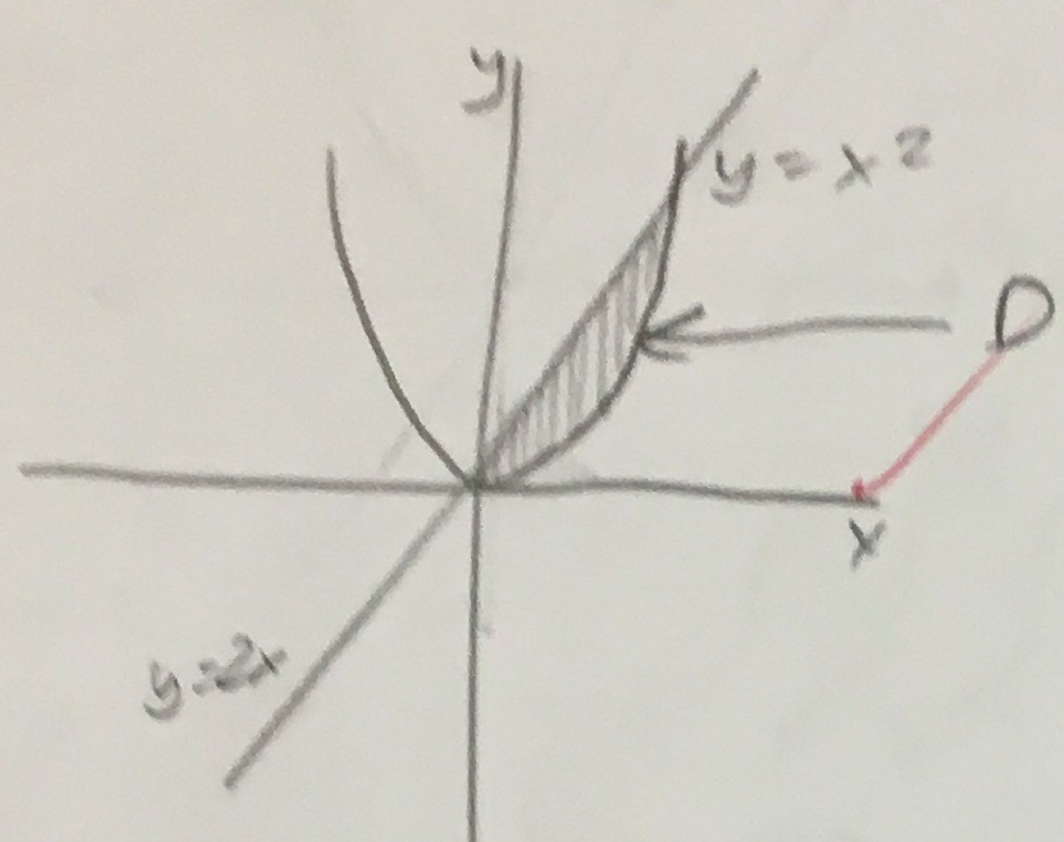
or

$\displaystyle\int_0^4\int_{y/2}^{\sqrt{y}} \frac{x}{4-y}\,dx\,dy$

$\displaystyle\frac{1}{2}\int_0^4 \left(\frac{1}{4-y}\right)\left(y - \frac{y^2}{4}\right) dy$

$\left(\dfrac{y}{4-y}\right), - \dfrac{y^2}{16-4y}$

a)



b)

Either

$\displaystyle\int_0^2\int_{x^2}^{2x} \frac{x}{4-y}\,dy\,dx$

vertically simple

Or

$\displaystyle\int_0^4\int_{y/2}^{\sqrt{y}} \frac{x}{4-y}\,dx\,dy$

horizontally simple

(I couldn't solve either)

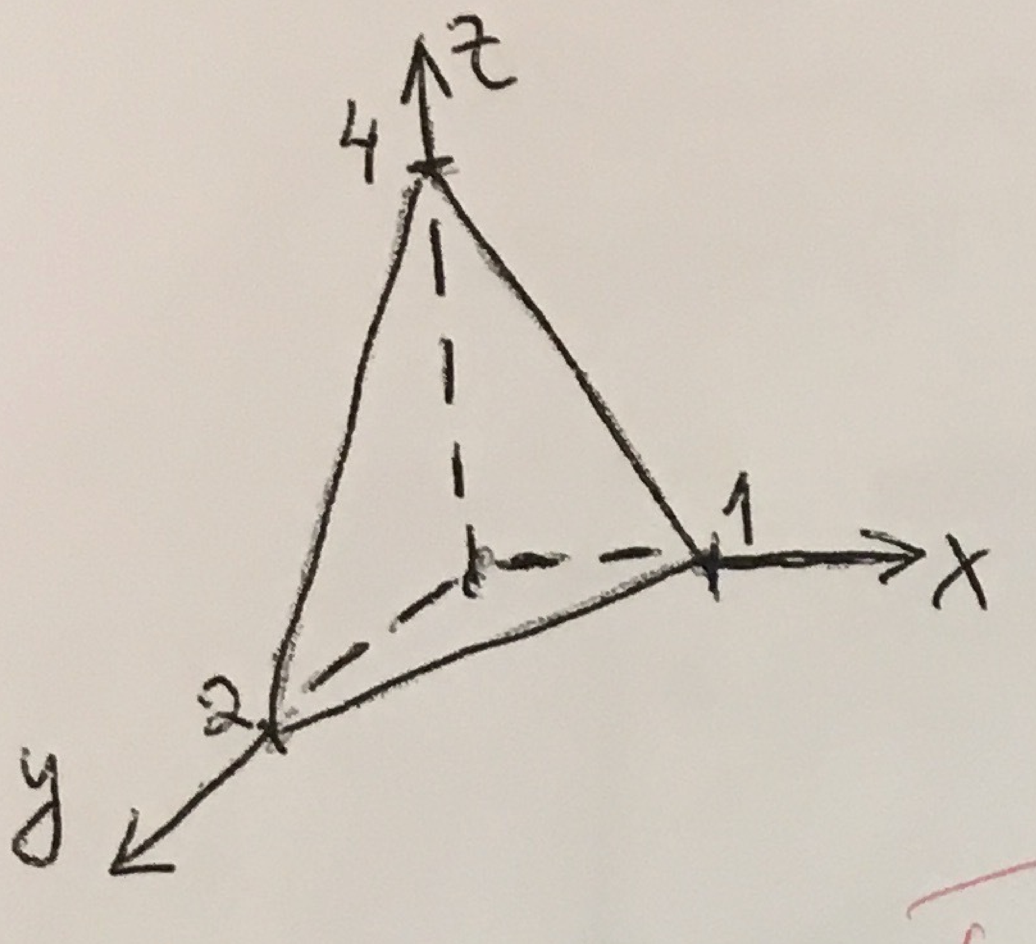
$\displaystyle\int_0^2 x\ln(4-y)\Big|_{x^2}^{2x}\,dx$

$$\boxed{\int_0^2 x\ln(4-2x) - x\ln(4-x^2)\,dx}$$

$$\boxed{\frac{1}{2}\int_0^4\int_{y/2}^{\sqrt{y}} \left(\frac{1}{4-y}\right)\left(y - \frac{y^2}{4}\right) dy}$$

?

Problem 3. (20 points) Let W be the tetrahedron in the first octant $x, y, z \geq 0$ with vertices at the points $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,4)$ (see the figure). Evaluate the triple integral of the function $f(x,y,z) = 1/(1-x)$ over W.



$x, y, z \geq 0$

$f(x,y,z) = 1/1-x$

want x on the inside

$x = \frac{4-z-2y}{4}$

plane equation $= 4x + 2y + z = 4$

$2y + z = 4$

$y = \frac{4-z}{2}$

$$\int_0^4 \int_0^{\frac{4-z}{2}} \int_0^{\frac{4-z-2y}{4}} (1-x)^{-1} \, dx \, dy \, dz$$

✓

$\ln 1 = x$

$e^x = 1$

$x = 0$

$$-\int_0^4 \int_0^{\frac{4-z}{2}} \ln(1-x) \Big|_0^{\frac{4-z-2y}{4}} dy\, dz$$

$$-\int_0^4 \int_0^{\frac{4-z}{2}} \ln(1-4+z+2y) - \ln 1 \, dy\, dz$$

$2 - \frac{z}{2}$

integral errors

$$-\int_0^4 \int_0^{\frac{4-z}{2}} \ln(2y+z-3)\, dy\, dz = -\int_0^4 (2y+z-3)\ln(2y+z-3) - (2y+z-3) \Big|_0^{\frac{4-z}{2}} dz$$

-5

$\int \ln x \, dx = ?$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x} \quad v = x$

$\int u\,dv = \ln(x)x - \int 1$

$\int \ln x\,dx = x\ln(x) - x$

$$= -\int_0^4 (1)\ln(1) - (1) - (z-3)\ln(z-3) + (z-3)\, dz$$

$$= \int_0^4 z - 4 - (z-3)\ln(z-3)\, dz$$

$\int x \ln x \, dx$

$u = x \quad dv = \ln x$

$du = 1 \quad v = x$

$\int x\ln x = x^2 - \int x$

$\int x \ln x = x^2 - \frac{1}{2}x^2$

$= \frac{1}{2}x^2$

$$= \left(\frac{1}{2}z^2 - 4z - \frac{1}{2}(z-3)^2\right)\Big|_0^4$$

$$= -(8 - 16 - \frac{1}{2} + \frac{9}{2}) \longrightarrow -(8-16+4) = -(-4) = \boxed{4}$$

$8 - 16 + 4 = -(-4) = 4$

Problem 4. (20 points) Let W be the region bounded by the sphere $x^2+y^2+z^2=9$ and (above) the cone $z=\sqrt{x^2+y^2}$. Find the volume of W using spherical coordinates.

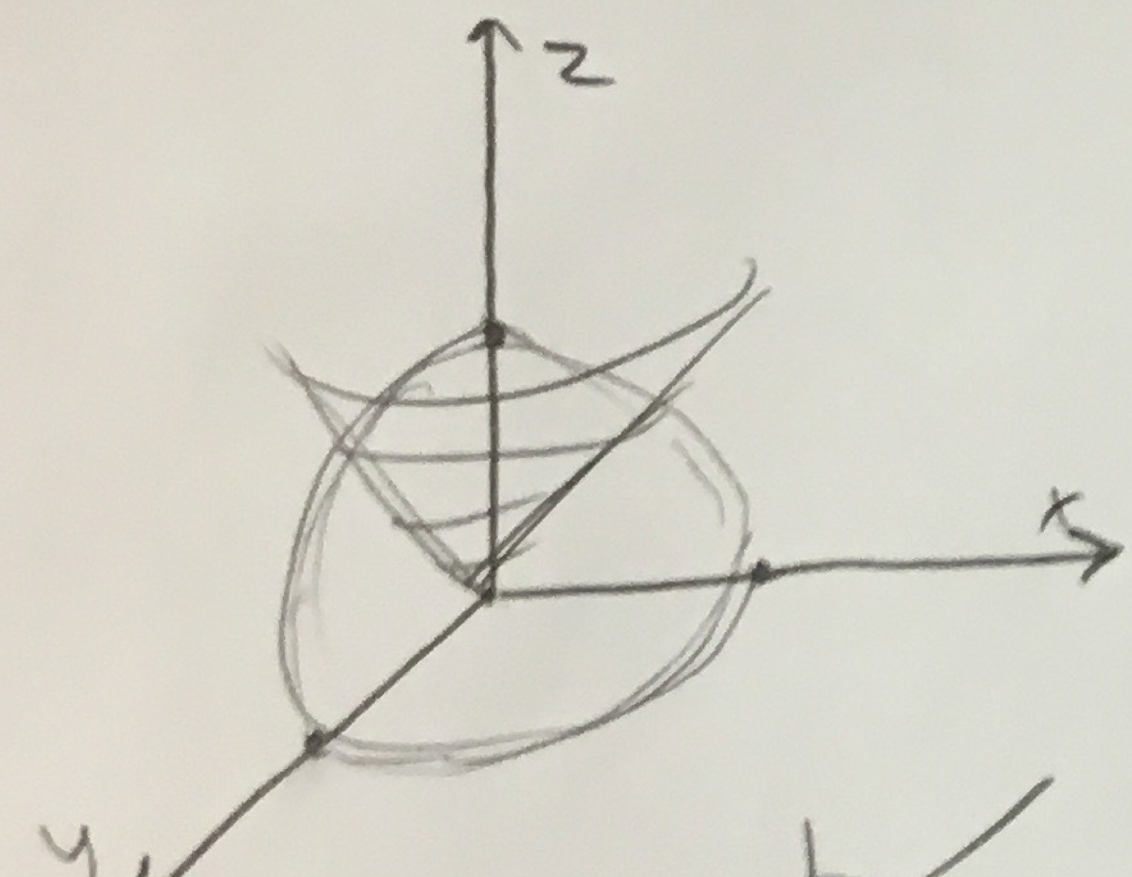
$r = \rho\sin\varphi$ $\qquad \rho^2\sin\varphi$

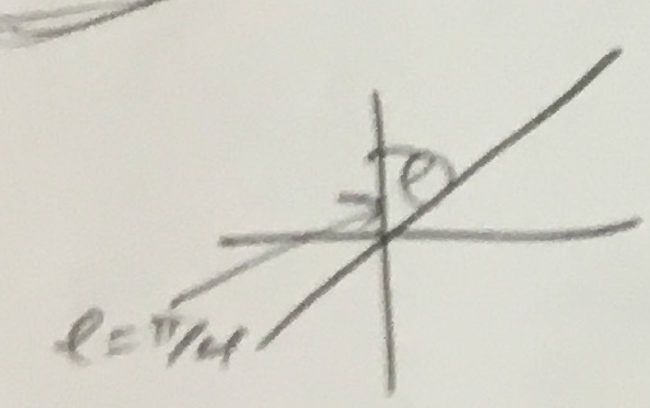
$z = \rho\cos\varphi$

$x^2+y^2+z^2 = 9 \Rightarrow r = 3$

$z = \sqrt{x^2+y^2}$

find the volume





$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} (1)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \rho^3\Big|_0^3 \sin\varphi\, d\varphi\, d\theta$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} 27\sin\varphi\, d\varphi\, d\theta$$

$$9\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi/4}\, d\theta$$

$$9\int_0^{2\pi} -\frac{\sqrt{2}}{2}+1\, d\theta$$

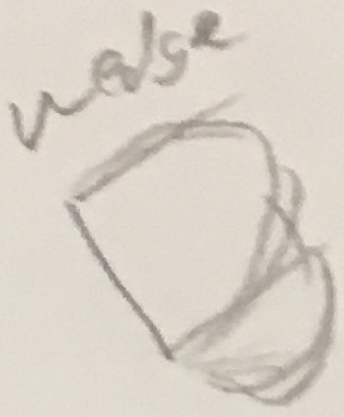
$$\boxed{18\pi\left(1-\frac{\sqrt{2}}{2}\right) \text{ units}^3}$$

Problem 5. (20 points) Let W be the region bounded by the cylinder $x^2+y^2=1$ and two half-planes $x=|z|$.

(a) (10 points) Find the volume of W.

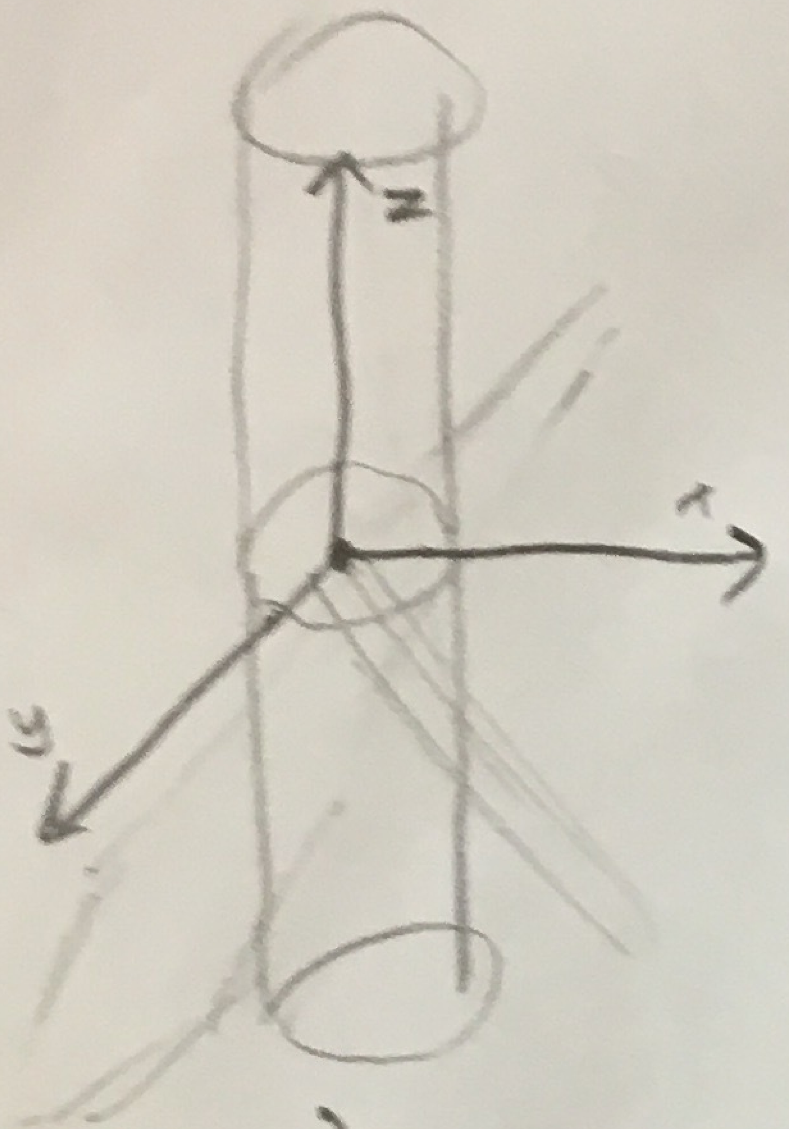
(b) (10 points) Find the centroid of W (i.e. the center of mass assuming the mass density $\delta(x,y,z)=1$) using cylindrical coordinates.

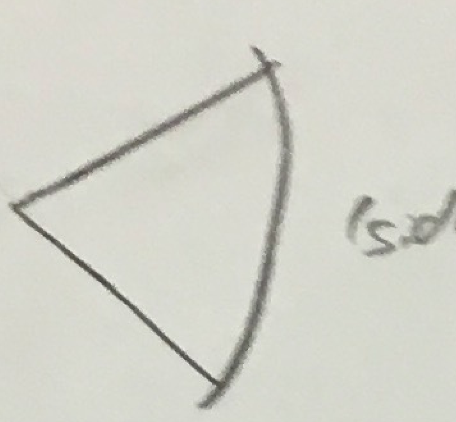
wedge



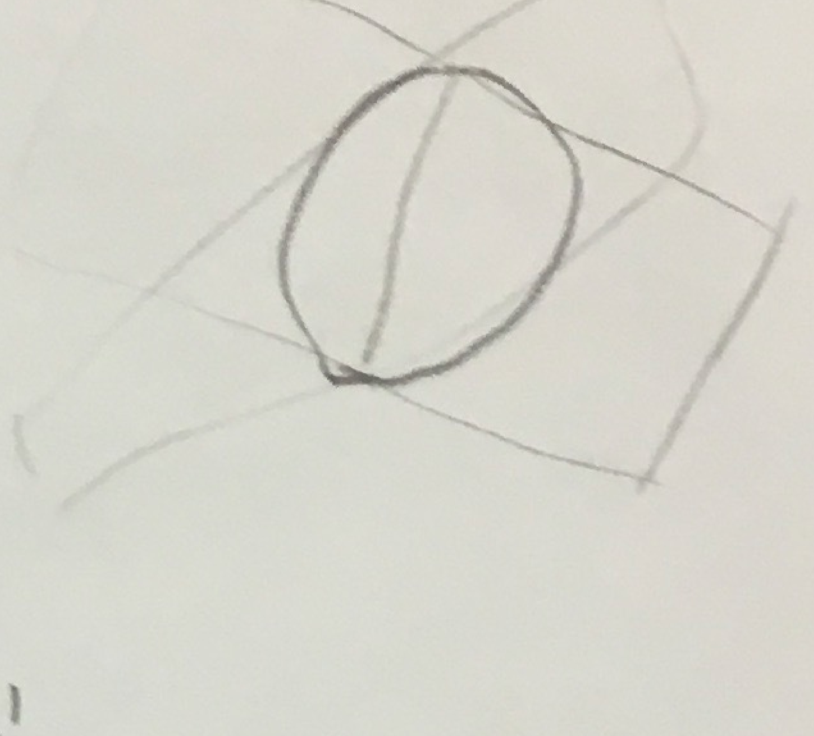
$x^2+y^2=1$ $\qquad x=|z|$

$\hookrightarrow x=-z,\ x=z$





(side)



a)

$$\text{vol(wedge)} = \int_{-\pi/2}^{\pi/2}\int_0^1 x-(-x)\; r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 2r^2\cos\theta\, dr\,d\theta = \frac{2}{3}\int_{-\pi/2}^{\pi/2} r^3\cos\theta\Big|_0^1 d\theta$$

$$= \frac{2}{3}\sin\theta\Big|_{-\pi/2}^{\pi/2} \to \frac{2}{3}(1-(-1)) = \boxed{\frac{4}{3}\text{units}^3}$$

✓

b)

$$X_{cm} = : \frac{\iiint x\,\delta(r,y,z)}{\iiint \delta(r,y,z)} = \frac{\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-x}^{x} x(1)\,dz\,r\,dr\,d\theta}{4/3}$$

$4/3 \longrightarrow \iiint \delta(x,y,z)$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r\cos\theta}^{r\cos\theta} r^2\cos\theta\, dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 2r^3\cos^2\theta\, dr\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} r^4\cos^2\theta\Big|_0^1 = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2\theta$$

see work on back of page $\qquad X_{cm} = \frac{\pi}{4}$

$$Y_{cm} = \frac{\iiint y\,\delta(x,y,z)}{\iiint \delta(x,y,z)} \to \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-x}^{x} y(1)\,dz\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r\cos\theta}^{r\cos\theta} r\sin\theta\,dz\,r\,dr\,d\theta$$

$4/3 \to \iiint \delta(x,y,z)$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 (r^3\cos\theta\sin\theta + r^3\cos\theta\sin\theta)\,dr\,d\theta = \frac{2}{4}\int_{-\pi/2}^{\pi/2}(r^4\cos\theta\sin\theta)\Big|_0^1 d\theta \to \frac{1}{2}\int_{-\pi/2}^{\pi/2}\frac{1}{2}\sin 2\theta\, d\theta = \frac{1}{4}\left(\frac{-\cos 2\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2}$$

$\downarrow$

$\boxed{0}$

$\boxed{Y_{CM}=0}$

$\boxed{\text{Centroid } \langle \frac{\pi}{4}, 0 \rangle}$

$z = ?$