20W-MATH32B-3 Midterm 2

AN LE

TOTAL POINTS

45 / 50

QUESTION 1

1 Q1 6 / 6

- ✓ **- 0 pts** **Correct**
- **- 3 pts** Incorrect Formula for divergence.
- **- 1 pts** Incorrectly calculated the divergence at P. (This includes not putting the point into the divergence and finding the value at the point)
- **- 2 pts** Incorrectly determined the behavior at point P. If it is understood that a negative div means a contraction, then this will not be taken away, but this understanding must be clear.

QUESTION 2

2 Q2 6 / 6

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Incorrect parameterisation.
- **- 2 pts** Incorrect formula for scalar line integral.
- **- 1 pts** Incorrect magnitude of derivative.
- **- 1 pts** Incorrect final answer.

QUESTION 3

3 Q3a 6 / 6

- **+ 3 pts** Correct bounds
- ✓ **+ 3 pts** **Correct bounds for upper half (note: these bounds may affect answer in (b))**
- **+ 2 pts** One bound incorrect
- **+ 1 pts** Two or more bounds incorrect
- ✓ **+ 3 pts** **Correct integration**
- **+ 2 pts** Small error in integration
- **+ 1 pts** Some attempt at integration
- **+ 1 pts** Some attempt
- **+ 0 pts** No attempt
- **+ 6 pts** Correct using volumes

QUESTION 4

4 Q3b 8 / 8

- ✓ **+ 2 pts** **x_CM = 0**
- ✓ **+ 2 pts** **y_CM = 0**
- ✓ **+ 2 pts** **Correct set up for z integral**
- **+ 1 pts** Incorrect set up for z integral
- ✓ **+ 1 pts** **Correct integration**
- ✓ **+ 1 pts** **Divided by mass from (a)**
- **+ 2 pts** said something about symmetry (but didn't correctly identify x_cm and y_cm)
- **+ 1 pts** Some attempt
- **+ 0 pts** No attempt

QUESTION 5

5 Q4 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Calculation error
- **- 2.5 pts** Integration error / no integration
- **- 2.5 pts** Didn't Include Jacobian in Integral
- **- 2.5 pts** Wrong Jacobian
- **- 5 pts** Didn't change variables correctly
- **- 10 pts** No significant progress

QUESTION 6

6 Q5 5 / 5

- ✓ **- 0 pts** **Correct**
- **- 5 pts** Incorrect / unrelated
- **- 1 pts** end and start backwards
- **- 1 pts** Minor mistakes (like integrand missing dot product, or written a to b)
- **- 1 pts** Unclear/incorrect points' relationship with curve, or implied a straight line
- **- 1 pts** Int F = f(end)-f(start), said F conservative, but didn't say that f was the potential
- **- 3 pts** Int F = f(end) - f(start), no mention of F being conservative or relationship between f and F
- **- 3 pts** Fundamental mistakes, but correct format

- **2 pts** Wrote double/triple integral, but understood that it is along a curve
- **1 pts** Wrote extra incorrect/unrelated things
- **2 pts** F=f'
- **2 pts** Int F = Int f(end)-f(start)
- **2 pts** Wrote a scalar version with f'
- **4 pts** F = grad f = f(end)-f(start)
- **2 pts** The value of what integral?
- **1 pts** F(end)-F(start) (instead of f)
- **2 pts** Path independence claimed, but no formula
- **2 pts** Unclear logic flow

QUESTION 7

7 Q6 3 / 3

✓ **- 0 pts Correct**

- **3 pts** Incorrect

QUESTION 8

8 Q7 1 / 6

- **0 pts** Correct
- **3 pts** Curl F = 0 claimed with no justification
- **2 pts** Logic flow unclear

✓ **- 5 pts Did not mention Curl F = 0 or a potential f**

- **4 pts** Conservative claimed with no justification
- **1 pts** Did not mention domain simply connected, or did not use properly
- **1 pts** Checked example only, but claimed Curl F = 0 in general
- **1 pts** Small mistakes / wrote extra things that are incorrect
- **4 pts** Curl F = 0 or F conservative with faulty justification
- **6 pts** Blank / nothing helpful
- **1 pts** Showed F conservative, but did not conclude properly
- **3 pts** "F_i can be written as d/dx_i"
- **3 pts** Checked example only, no claim on general case
- **4 pts** Checked Curl F = 0 only, nothing else
- **4 pts** Closed curve -> int from t=a to t=a with no other justification

QUESTION 9

9 Bonus 0 / 0

✓ **+ 0 pts Nothing helpful**

+ **1 pts** Good starting point
+ **2 pts** Good ideas, but not complete
+ **3 pts** Correct

STUDENT NAME: An Le

STUDENT ID NUMBER:

DISCUSSION SECTION NUMBER: 3F

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	12	
3	6	
4	8	
5	10	
6	8	
7	6	
Total:	50	

1. [6 pts] Compute Div $\overrightarrow{\mathbf{F}}$ for the vector field

$$\overrightarrow{\mathbf{F}}(x,y,z) = \langle x^2y, \cos z, z + e^{xy} \rangle.$$

If $\overrightarrow{\mathbf{F}}$ is describing some sort of motion/flow of particles, is the overall flow near the point $P = (1,-2,3)$ outward (P is a source/point of expansion) or inward (P is a sink/point of contraction)?

$\text{div}\vec{F} = \langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \rangle \cdot \langle x^2y; \cos z, z + e^{xy} \rangle$

$= 2yx + 0 + 1 = \boxed{2yx+1 = \text{div}\vec{F}}$

$\text{div}\vec{F} @ P = (1,-2,3) \to 2(-2)(1) + 1 = -4 + 1 = -3$

Since div$\vec{F}$ at point $P = -3$ a negative number, the overall flow is $\boxed{\text{inward}}$ due to negative divergence. P is a $\boxed{\text{sink/point of contraction}}$

2. [6 pts] If C is the line segment connecting $(0,1)$ to $(1,0)$ in the xy-plane, compute the line integral $\int_C (y^2 + x^3)\, ds$ Scalar

$\vec{r}(t) = (1-t)\langle 0,1\rangle + t\langle 1,0\rangle = \langle 0, 1-t\rangle + \langle t, 0\rangle = \langle t, 1-t\rangle$

$\vec{r}'(t) = \langle 1, -1\rangle$ Line segment, $t=0 \to t=1$

$|\vec{r}'(t)| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\int_C (y^2 + x^3) = \int_{t=0}^{1} (1-t)^2 + (t)^3 \cdot \sqrt{2}\, dt$

$= \sqrt{2}\int_{t=0}^{1} t^2 - 2t + 1 + t^3\, dt$

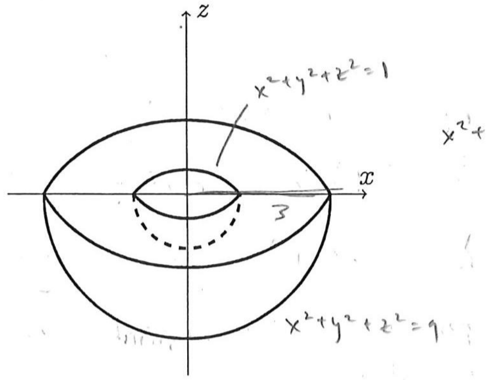
$= \sqrt{2}\left[\frac{1}{3}t^3 - t^2 + t + \frac{1}{4}t^4\right]_{t=0}^{1}$

$= \sqrt{2}\left[\frac{1}{3}(1)^3 \cancel{-1} \cancel{+1} + \frac{1}{4}(1)^4 - 0\right]$

$= \sqrt{2}\left[\frac{1}{3} + \frac{1}{4}\right]$ $\qquad \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

$= \sqrt{2}\left(\frac{7}{12}\right) = \boxed{\frac{7\sqrt{2}}{12}}$

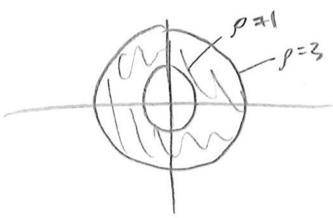
3. Consider a spherical cantaloupe of radius 3, centered at the origin, which is sliced in half. The 'top' half is discarded so that only the 'bottom' half (below the xy-plane) remains. Next, the inner seeds are scooped out leaving a half-spherical 'hole' of radius 1. See the figure below, where the y-axis points into the page (and has been omitted for clarity).



$x^2+y^2+z^2=9$

Total mass $= \iiint \delta(x,y,z)\, dV$

(a) [6 pts] Assuming the cantaloupe has constant mass-density function $\delta(x, y, z) = 6$, find the total mass of the remaining cantaloupe.



$\theta = 0 \to \theta = 2\pi$ $\quad z = 0 \to \rho\cos\phi = 0 \to \cos\phi = 0$

$\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=1}^{3} 6\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 6\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \sin\phi\left(\tfrac{1}{3}\rho^3\right)\Big|_{\rho=1}^{3} d\phi\, d\theta$$

$$= 6\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \sin\phi\left[\tfrac{1}{3}(27) - \tfrac{1}{3}\right) d\phi\, d\theta$$

$$= 6\left(9 - \tfrac{1}{3}\right)\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \sin\phi\, d\phi\, d\theta$$

$$= \overset{2}{\cancel{6}}\left(\frac{26}{\cancel{3}}\right)\int_{\theta=0}^{2\pi} -\cos\phi\,\Big|_{\phi=0}^{\pi/2} d\theta$$

$$= 52\int_{\theta=0}^{2\pi} \cancel{-\cos\left(\tfrac{\pi}{2}\right)} + \cos(0)\, d\theta$$

$$= 52\int_{\theta=0}^{2\pi} 1\, d\theta$$

$$= 52(\theta)\Big|_{\theta=0}^{2\pi}$$

$$= 52(2\pi - 0) = \boxed{104\pi}$$

$$\begin{array}{r} 52 \\ \times\ 2 \\ \hline 104 \end{array}$$

(b) [8 pts] Again assuming that our cantaloupe had constant density $\delta(x, y, z) = 6$, find the (x, y, z)-coordinates of its center of mass. *(HINT: You can use symmetry arguments to determine two of the three coordinates.)*

centered at the origin, so symmetrical about the x- and y planes, thus x_{cm} and $y_{cm} = 0$

$$z_{cM} = \frac{\iiint z\,\delta(x,y,z)\,dV}{\iiint \delta(x,y,z)\,dV} \qquad z = \rho\cos\phi$$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=1}^{\rho=3} 6\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 6\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\cos\phi\sin\phi\int_{\rho=1}^{\rho=3}\rho^3\,d\rho\,d\phi\,d\theta$$

$$= 6\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\cos\phi\sin\phi\left[\frac{1}{4}\rho^4\right]_1^3 d\phi\,d\theta \qquad \frac{1}{4}(3^4-1) = \frac{81}{4}-\frac{1}{4} = \frac{80}{4} = 20$$

$$= 6(20)\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta$$

$u = \sin\phi$

$du = \cos\phi\,d\phi$

$$= 120\int_{\theta=0}^{2\pi}\int_0^1 u\,du\,d\theta$$

$$= 120\int_{\theta=0}^{2\pi}\frac{1}{2}u^2\Big|_{u=0}^{1}d\theta$$

$$= 120\int_{\theta=0}^{2\pi}\frac{1}{2}(1^2-0^2)\,d\theta$$

$$= 60\int_{\theta=0}^{2\pi}1\,d\theta = 60[\theta]_0^{2\pi}$$

$$= 60(2\pi) = 120\pi$$

$8\overline{)120}$ = 15; -8, 40 $\qquad$ $8\overline{)104}$ = 13; -8, 24

$$z_{cm} = \frac{120\pi}{104\pi} \overset{\div 2}{=} \frac{60}{52} \overset{\div 2}{=} \frac{30}{26} = \frac{15}{13}$$

$$\boxed{\text{center of mass} = \left(0, 0, -\frac{15}{3}\right)}$$

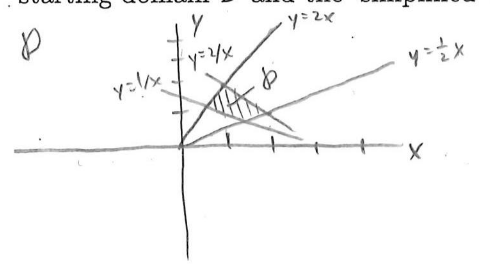
x y z

$-\frac{15}{13}$ because cantaloupe is below xy-plane (z must be negative)

4. [10 pts] Let $\mathcal{D}$ be the domain in the first quadrant of the xy-plane bounded by the following four curves:

$$y = 2x, \quad y = \frac{1}{2}x, \quad y = \frac{2}{x}, \quad y = \frac{1}{x}.$$

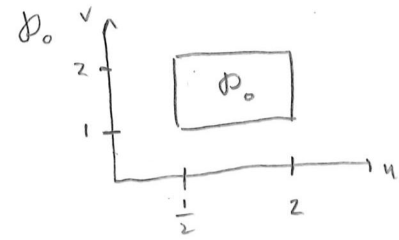
Use a change of variables to compute $\iint_{\mathcal{D}} \frac{2\pi^2 y}{x} \sin(\pi xy) \cos\left(\pi \frac{y}{x}\right) dA$. Be sure to draw both the starting domain $\mathcal{D}$ and the 'simplified' domain $\mathcal{D}_0$ resulting from your change of variables.



$$\xrightarrow{F}$$

$u = \frac{y}{x} = yx^{-1}$

$v = xy$



$y = 2x$, $y = \frac{1}{2}x$ $\rightarrow$ $y/x = 2$, $y/x = \frac{1}{2}$ $\rightarrow$ $u = \frac{1}{2}$, $u = 2$ | $y = 2/x$, $y = 1/x$ $\rightarrow$ $xy = 2$, $xy = 1$ $\rightarrow$ $v = 1$, $v = 2$

$$\text{Jac}F = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} = \begin{vmatrix} -yx^{-2} & \frac{1}{x} \\ y & x \end{vmatrix} = \frac{-y\cancel{x}}{x^{\cancel{2}}} - \frac{y}{x} = \left|\frac{-2y}{x}\right| = \frac{2y}{x} = 2u$$

$$\int_{v=1}^{2} \int_{u=1/2}^{2} 2\pi^2 (u) \sin(\pi v) \cos(\pi u) \left|\frac{1}{2u}\right| du\, dv \quad \leftarrow \iint f \left|\frac{1}{\text{Jac}F}\right| dA$$

$$= \cancel{2}\pi^2 \int_{v=1}^{2} \int_{u=1/2}^{2} \frac{1}{\cancel{2}} \sin(\pi v) \cos(\pi u)\, du\, dv$$

$$= \pi^2 \int_{v=1}^{2} \sin(\pi v) \left[\frac{1}{\pi} \sin(\pi u)\right]\Big|_{u=1/2}^{2} dv$$

$$= \pi^2 \int_{v=1}^{2} \sin(\pi v) \left[\cancel{\frac{1}{\pi}\sin(2\pi)} - \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right)\right] dv$$

$$= \pi^{\cancel{2}} \int_{v=1}^{2} \sin(\pi v) \left[-\frac{1}{\cancel{\pi}}(1)\right] dv$$

$$= -\pi \int_{v=1}^{2} \sin(\pi v)\, dv$$

$$= -\pi \left[-\frac{1}{\pi} \cos(\pi v)\right]\Big|_{v=1}^{2}$$

$$= -\pi \left(-\frac{1}{\pi} \cos(2\pi) + \frac{1}{\pi} \cos(\pi)\right]$$

$$= -\pi \left[-\frac{1}{\pi}(1) + \frac{1}{\pi}(-1)\right]$$

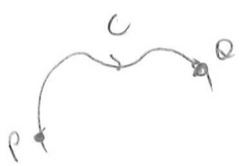
$$= -\pi \left[-\frac{2}{\pi}\right]$$

$$= \frac{2\pi}{\pi} = \boxed{2}$$

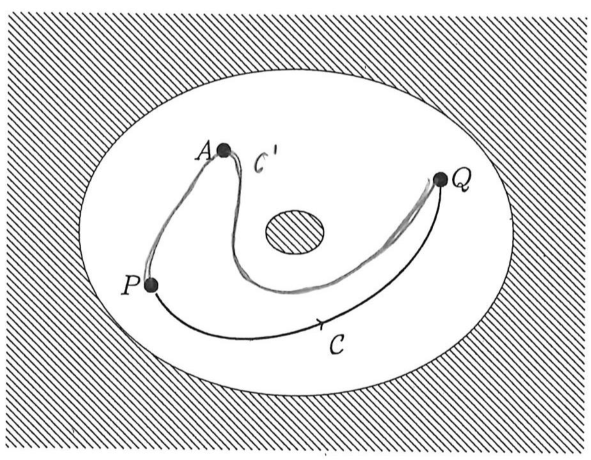
5. [5 pts] The Fundamental Theorem of 1-Variable Integration states that $\int_a^b f'(x)dx = f(b) - f(a)$. State the corresponding Fundamental Theorem of Line Integration. Be very precise!

$$\int_C \vec{F}\cdot d\vec{r} = f(\text{end}) - f(\text{start}) = f(Q) - f(P)$$

$\vec{F}$ must be conservative (there exists a potential function f for $\vec{F}$ such that $\vec{F} = \nabla f$). $\vec{F}$ is conservative and thus path-independent if curl $\vec{F} = 0$ over an open, simply connected domain. C is a curve that travels from P to Q, for $f(Q) - f(P)$.



6. [3 pts] Suppose that $\text{Curl}(\vec{\mathbf{F}}) = \vec{0}$ throughout the domain pictured below. Draw a curve C' from P to Q that passes through the point A, but also guarantees that $\int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.



7. [6 pts] Consider a vector field $\overrightarrow{\mathbf{F}}(x, y, z)$ that has the form

$$\overrightarrow{\mathbf{F}}(x, y, z) = \langle F_1(x), F_2(y), F_3(z) \rangle ,$$

that is, the first component is only a function of x, the second component is only a function of y, and so forth. An example would be something like $\langle x^3, y + e^{\cos y}, \ln z \rangle$.

Assuming that the domain $\mathcal{D}$ of $\overrightarrow{\mathbf{F}}$ is simply connected, prove that $\oint_C \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = 0$ for any closed curve $\mathcal{C}$ in $\mathcal{D}$.

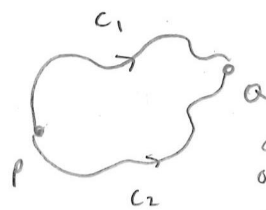
closed curve C

$C = C_1 - C_2$ $\longrightarrow$ $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$

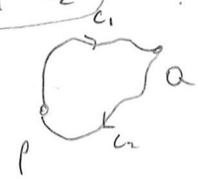
$= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}$

$= \int_P^Q \vec{F} \cdot d\vec{r} + \int_Q^P \vec{F} \cdot d\vec{r}$

$\boxed{\int_C \vec{F} \cdot d\vec{r} = 0}$



$\longrightarrow$ change orientation of C_2 so that it loops back around in closed curve



(BONUS: Can you explain why the format of $\overrightarrow{\mathbf{F}}$ above actually guarantees that $\mathcal{D}$ is simply connected? This is tricky, do not spend time on this if you have not finished all of the other questions on the exam.)

simply connected = no holes in 2D

no cylinder or shape removed from edge to edge in 3D (soccer ball)

$\text{curl}\, \vec{F} = 0$ $\quad \langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \rangle \times \langle F_1(x), F_2(y), F_3(z) \rangle = 0$

↳ $\vec{F}$ is conservative, path independent for simply connected, open domain

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