MATH 32B — Midterm I — Jan. 27, 2029

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STUDENT ID NUMBER: ______

DISCUSSION SECTION NUMBER: ______

Directions

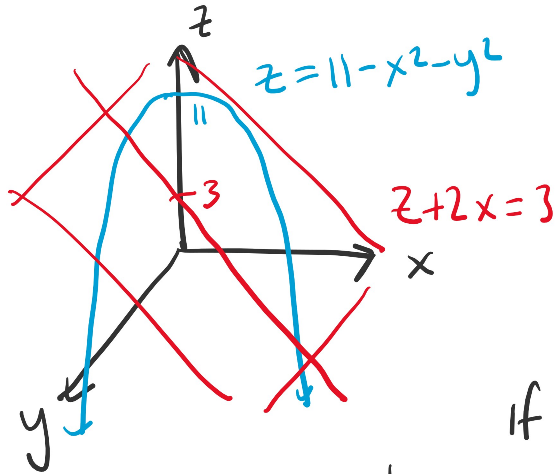
Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	8	
3	10	
4	12	
5	10	
6	10	
Total:	50	

Note: I marked Question 3. I havn't checked my answers for the other questions so they might be wrong. Let me know if you find any mistakes.

1. [8 pts] Consider the function $f(x, y, z) = x+y+z$. Set up (BUT DO NOT SOLVE) an iterated triple integral to find $\iiint_{\mathcal{W}} f\,dV$ where $\mathcal{W}$ is the 3-dimensional region enclosed by $z = 11-x^2-y^2$ and $z + 2x = 3$. Include a rough sketch of both $\mathcal{W}$ and its 2-dimensional projection $\mathcal{D}$ used to find the various bounds of integration (your sketch of $\mathcal{D}$ will affect your grade).



I want to use the projection onto the x-y plane.

To figure this out, I need to figure out the intersection.

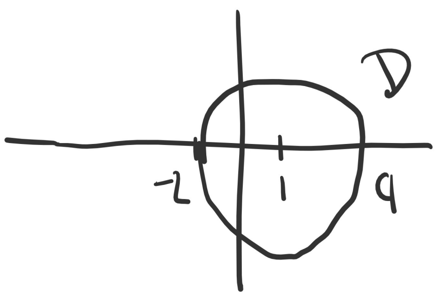
If $z = 11-x^2-y^2$ and $z = 3-2x$

then $3-2x = 11-x^2-y^2$

$y^2+x^2-2x = 8$

$y^2+(x-1)^2-1 = 8$

$y^2+(x-1)^2 = 9.$

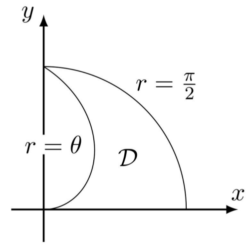


Hence, the projection D will be a circle centred at (1,0) with radius 3.

Since z goes from the plane $z=3-2x$ to the paraboloid $z=11-x^2-y^2$, we get that the integral is:

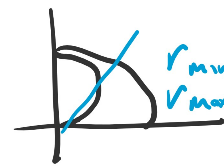
$$\iiint_W x+y+z\,dV = \iint_D \int_{3-2x}^{11-x^2-y^2} x+y+z\,dV = \int_{-2}^{4}\int_{-\sqrt{9-(x-1)^2}}^{\sqrt{9-(x-1)^2}}\int_{3-2x}^{11-x^2-y^2} x+y+z\,dV$$

2. Consider the 'tooth-shaped' domain $\mathcal{D}$ in the first quadrant shown below.



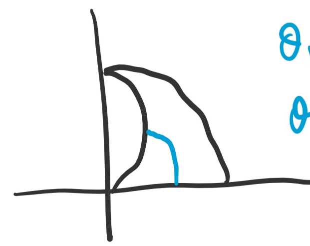
(a) [6 pts] Write $\iint_{\mathcal{D}} 4r\,dA$ as an iterated polar integral in two ways, one for integrating r first and the other for integrating θ first.

$$\iint_D 4r\,dA = \int_0^{\pi/2}\int_\theta^{\pi/2} 4r^2\,dr\,d\theta$$



$r_{min} = \theta$
$r_{max} = \pi/2$

$$= \int_0^{\pi/2}\int_0^{r} 4r^2\,d\theta\,dr$$



$\theta_{min} = 0$
$\theta_{max} = r$.

(b) [4 pts] Compute $\iint_{\mathcal{D}} 4r\,dA$ using either of your iterated integrals above.

$$\int_0^{\pi/2}\int_0^{r} 4r^2\,d\theta\,dr = \int_0^{\pi/2} 4r^3\,dr = r^4\Big|_0^{\pi/2} = \frac{\pi^4}{16}$$

Note: We technically only "integrate in rectangular coordinates". This will be explained when we do change of variables. Essentially $\iint f(x,y)\,dA = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ changes the function to rectangular coordinates in the r-θ plane and $dA = r\,dr\,d\theta$ accounts for the dilation. So above, $\iint_D r\,dA$ means $\iint_D \sqrt{x^2+y^2}\,dA$ in less confusing notation.

3. [12 pts] Consider the function $f(x, y, z) = ye^{xy} + z$. Prove that there exists *some* point $P = (a, b, c)$ in the rectangular box domain $\mathcal{R} = [0, 1] \times [0, 1] \times [0, 2]$ where $f(P) = e - 1$. Quote the theorem you are using, and don't forget the hypotheses!

$$\bar{f} = \frac{\int_0^2 \int_0^1 \int_0^1 ye^{xy} + z \, dx\, dy\, dz}{\mathrm{vol}(R)}$$

$$= \frac{1}{2} \int_0^2 \int_0^1 e^{xy} + zx \Big|_0^1 dy\, dz$$

$$= \frac{1}{2} \int_0^2 \int_0^1 e^y + z - 1 \, dy\, dz$$

$$= \frac{1}{2} \int_0^2 e^y + zy - y \Big|_0^1 dz$$

$$= \frac{1}{2} \int_0^2 e + z - 2 \, dz$$

$$= \frac{1}{2} \left((e-2)z + \frac{z^2}{2} \Big|_0^2 \right)$$

$$= \frac{1}{2} \left(2(e-2) + 2 \right)$$

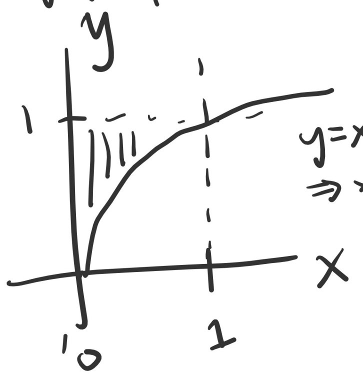
$\therefore \bar{f} = e - 1.$

Now, f is continuous, and R is a closed, bounded and connected domain. Hence the mean value theorem implies there exists a point $P \in R$ s.t. $f(P) = e - 1$.
Note, there are two more possible proofs which I'll include on the scratch paper.

4. [10 pts] Evaluate the double integral

$$\int_{x=0}^{x=1} \int_{y=x^{2/3}}^{y=1} 2xe^{y^4}\, dy dx.$$

We interchange order of integration.



$y = x^{2/3}$

$\Rightarrow x = y^{3/2}$

so the integral becomes:

$$\int_0^1 \int_0^{y^{3/2}} 2xe^{y^4} dx dy$$

$$= \int_0^1 x^2 e^{y^4} \Big|_0^{y^{3/2}} dy = \int_0^1 y^3 e^{y^4} dy = \frac{1}{4} e^{y^4} \Big|_0^1$$

$$= \frac{1}{4}(e-1)$$

5. [4 pts] Let $\mathcal{D}$ be the unit disc. Prove that

$$\pi \leq \iint_{\mathcal{D}} (2 + \cos(x^2y^3))dA \leq 3\pi.$$

We have $-1 \leq \cos(x^2y^3) \leq 1$ for all $(x,y) \in D$

so $1 \leq 2 + \cos(x^2y^3) \leq 3.$

Hence $\iint_D 1 \, dA \leq \iint_D (2+\cos(x^2y^3))dA \leq \iint_D 3 \, dA$

$$\pi \leq \iint_D 2+\cos(x^2y^3) \, dA \leq 3\pi$$

6. [6 pts] Suppose I have a continuous function $f(x, y)$ that is defined throughout the plane $\mathbb{R}^2$. Consider two domains: $\mathcal{D}_1$ is the unit disc at the origin, and $\mathcal{D}_2$ is the disc of radius 2 at the origin; in particular, $\mathcal{D}_1$ *is contained in* $\mathcal{D}_2$. Is it true that $\iint_{\mathcal{D}_1} f dA \leq \iint_{\mathcal{D}_2} f dA$? If so, explain why. If not, explain what extra hypothesis is needed to make it true.

No. Observe that since

$$\iint_{D_2} f \, dA = \iint_{D_2 \setminus D_1} f \, dA + \iint_{D_1} f \, dA$$

Then the inequality is equivalent to $\iint_{D_2 \setminus D_1} f \, dA \geq 0$. (*)

So for the inequality to be true, we must have (*) to be true. Any function where (*) is not true is a counterexample.

e.g. Take $f(x,y) = -1$. Then $\iint_{D_1} f \, dA = -\pi$

and $\iint_{D_2} f \, dA = -4\pi$ and $-4\pi < -\pi$.

Question 3:

proof 2: Notice that $f(1,1,2) = e+2 > e-1$ and $f(0,0,0) = 0 < e-1$. Since R is path connected, there exists a continuous path $\gamma: [0,1] \to R$ that connects these points. ie $\gamma(0) = (0,0,0)$ and $\gamma(1) = (1,1,2)$. Now, as f continuous on R, IVT on $f(\gamma(t))$ gives that there exists $c \in [0,1]$ s.t $f(\gamma(c)) = e-1$. ie $p = \gamma(c)$ is the required pt.

proof 3: observe $(0,0,e-1) \in R$ and $f(0,0,e-1) = e-1$.