MATH 32B Midterm I Practice Jan. 27, 2029

STUDENT NAME: SOLUTIONS

STUDENT ID NUMBER: __________

DISCUSSION SECTION NUMBER: __________

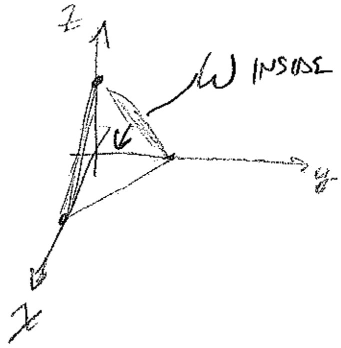
Directions

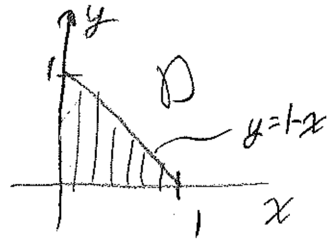
Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	8	
3	8	
4	8	
5	8	
6	12	
7	6	
Total:	50	

1. [8 pts] Compute the integral of the function $f(x,y,z) = e^z$ in the region $\mathcal{W}$ that is in the first octant but below the plane $x + y + z = 1$.





$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} e^{z}\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left(e^{1-x-y} - 1\right) dy\,dx$$

$$= \int_{x=0}^{1} -e^{1-x-y}\Big|_{y=0}^{1-x} - y\Big|_{y=0}^{1-x}\,dx$$

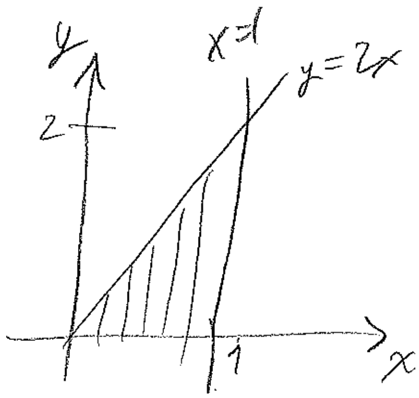
$$= \int_{x=0}^{1} -e^{0} + e^{1-x} - (1-x)\,dx$$

$$= \int_{x=0}^{1} e^{1-x} - 2 + x\,dx$$

$$= -e^{1-x}\Big|_{x=0}^{1} - 2x\Big|_{x=0}^{1} + \frac{1}{2}x^2\Big|_{x=0}^{1}$$

$$= -1 + e - 2 + \frac{1}{2} = e + \frac{1}{2} - 3$$

2. [8 pts] Evaluate the integral



$$\int_{y=0}^{y=2}\int_{x=y/2}^{x=1}\cos\left(\frac{\pi}{6}x^2\right)\,dxdy.$$

$$\| $$

$$\int_{x=0}^{1}\int_{y=0}^{y=2x}\cos\left(\frac{\pi}{6}x^2\right)dy\,dx$$

$$\| $$

$$\int_{x=0}^{1} y\cos\left(\frac{\pi}{6}x^2\right)\Big|_{y=0}^{2x}\,dx$$

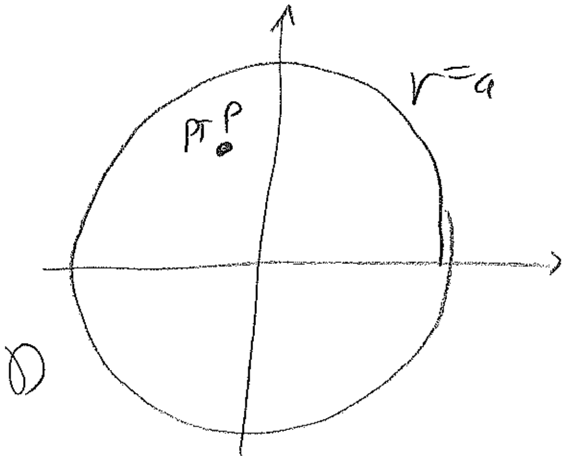
$$=\int_{x=0}^{1} 2x\cos\left(\frac{\pi}{6}x^2\right)dx$$

$$=\frac{6}{\pi}\sin\left(\frac{\pi}{6}x^2\right)\Big|_{x=0}^{1}$$

$$=\frac{6}{\pi}\sin\frac{\pi}{6}-\frac{6}{\pi}\sin 0$$

$$=\frac{3}{\pi}$$

3. [8 pts] Prove that, if we consider a 2-dimensional circular disc of radius a (centered at the origin, say), then the average distance between a point on the disc and the center is $\frac{2a}{3}$.



USING POLAR COORDS

$f(P)$ = DISTANCE OF P FROM ORIGIN
$= r$

THUS AVG DIST IS $\overline{f}$

WHICH IS $\iint_D r\,dA \Big/ \text{AREA}(D)$

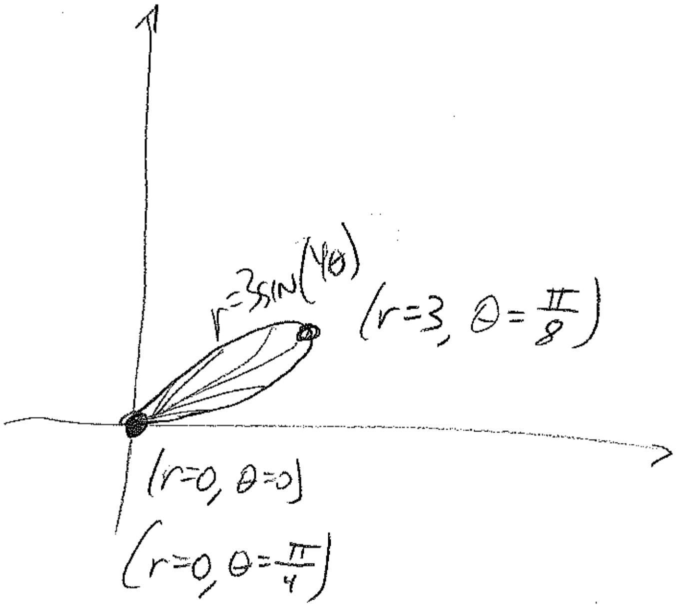
$$= \frac{\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\,r\,dr\,d\theta}{\pi a^2}$$

$$= \frac{1}{\pi a^2}\int_{\theta=0}^{2\pi} \frac{1}{3}r^3\Big|_{r=0}^{a} d\theta$$

$$= \frac{1}{\pi a^2}\int_{\theta=0}^{2\pi} \frac{a^3}{3}\,d\theta$$

$$= \frac{1}{\pi a^2}\cdot\frac{2\pi a^3}{3} = \frac{2a}{3}$$

4. [8 pts] Use a double integral to find the area inside one loop of the polar rose $r = 3\sin(4\theta)$. Include a reasonable sketch of the loop you are considering. *Hint: You may need the double-angle formula* $\sin^2(A) = \frac{1-\cos(2A)}{2}$.



$$\text{AREA} = \iint 1\,dA$$

$$= \int_{\theta=0}^{\pi/4} \int_{r=0}^{r=3\sin 4\theta} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{1}{2} r^2 \Big|_{r=0}^{3\sin 4\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{9}{2} \sin^2(4\theta)\,d\theta$$

$$= \frac{9}{2} \int_{\theta=0}^{\pi/4} \frac{1-\cos 8\theta}{2}\,d\theta$$

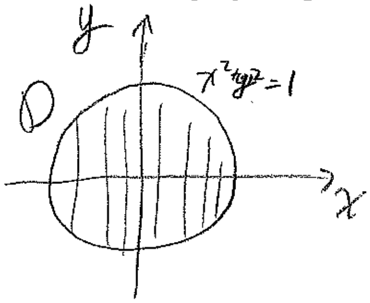
$$= \frac{9}{4} \int_{\theta=0}^{\pi/4} 1 - \cos 8\theta\,d\theta$$

$$= \frac{9}{4} \left(\theta - \frac{1}{8}\sin 8\theta\right)\Bigg|_{\theta=0}^{\pi/4}$$

$$= \frac{9\pi}{8}$$

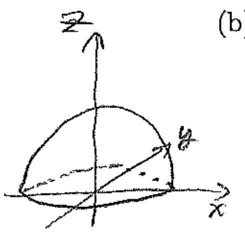
5. Consider the function $f(x, y, z) = xyz$ on the domain $\mathcal{W}$ which consists of the 'northern half of the unit ball' (that is, the set of points above the xy-plane but below the unit sphere).

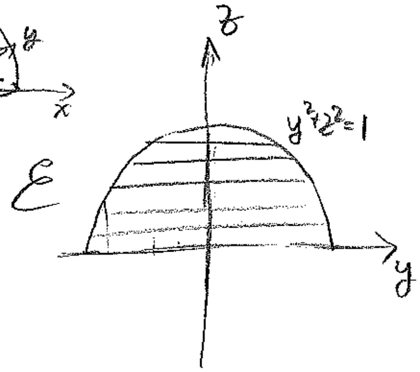
   (a) [4 pts] Set up (BUT DO NOT SOLVE) an iterated triple integral to find $\iiint_{\mathcal{W}} f dV$ integrating z first, then y, then x. Include a sketch of the 2-dimensional domain $\mathcal{D}$ coming from projecting $\mathcal{W}$.



$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=0}^{z=\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$$

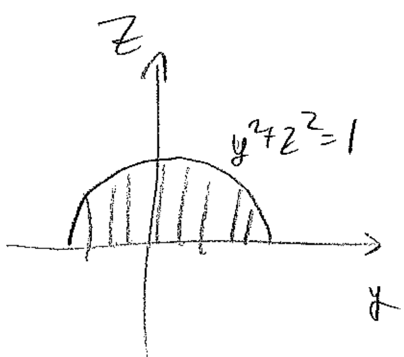
   (b) [4 pts] Repeat the previous part, but now integrate x first, then y, then z. Again, be sure to include a sketch of the 2-dimensional domain $\mathcal{E}$ coming from projecting $\mathcal{W}$.





$$\int_{z=0}^{1}\int_{y=-\sqrt{1-z^2}}^{y=\sqrt{1-z^2}}\int_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} xyz\, dx\, dy\, dz$$

   (c) [4 pts] Repeat one more time, but now integrate x first, then z, then y.



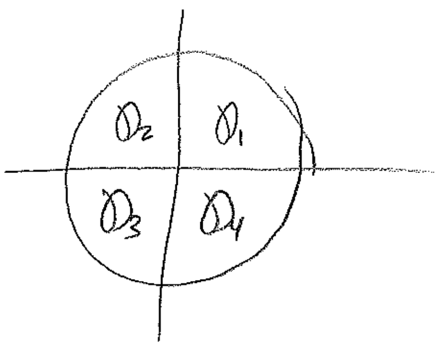
$$\int_{y=-1}^{1}\int_{z=0}^{z=\sqrt{1-y^2}}\int_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} xyz\, dx\, dz\, dy$$

6. [6 pts] Consider the following piecewise function

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ is in Quadrant I} \\ 2 & \text{if } (x,y) \text{ is in Quadrant II} \\ 3 & \text{if } (x,y) \text{ is in Quadrant III} \\ 4 & \text{if } (x,y) \text{ is in Quadrant IV} \end{cases}$$

What is the value of $\iint_{\mathcal{D}} f(x,y)dA$ when $\mathcal{D}$ is the unit disc centered at the origin? Explain. Note that, for the purposes of integration in cases such as this, you may disregard the values of f along the axes themselves.

BREAK $\mathcal{D}$ UP INTO 4 PIECES:



$$\iint_{\mathcal{D}} f\,dA = \iint_{\mathcal{D}_1} f\,dA + \iint_{\mathcal{D}_2} f\,dA + \iint_{\mathcal{D}_3} f\,dA + \iint_{\mathcal{D}_4} f\,dA$$

$$= \iint_{\mathcal{D}_1} 1\,dA + \iint_{\mathcal{D}_2} 2\,dA + \iint_{\mathcal{D}_3} 3\,dA + \iint_{\mathcal{D}_4} 4\,dA$$

$$= \text{AREA}(\mathcal{D}_1) + 2\,\text{AREA}(\mathcal{D}_2) + 3\,\text{AREA}(\mathcal{D}_3) + 4\,\text{AREA}(\mathcal{D}_4)$$

$$= \frac{\pi}{4} + 2\frac{\pi}{4} + 3\frac{\pi}{4} + 4\frac{\pi}{4}$$

$$= \frac{5\pi}{2}$$

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