STUDENT NAME: Solutions

STUDENT ID NUMBER: ______________________

DISCUSSION SECTION NUMBER: ______________________

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	10	
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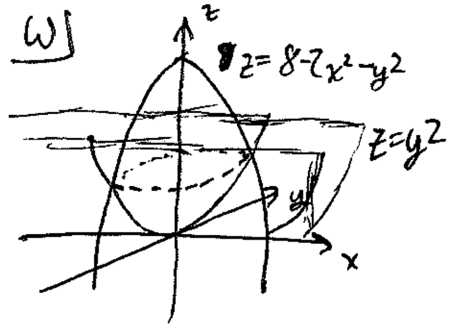
11	12	
Total:	120	

1. [10 pts] Integrate the function $f(x, y, z) = x$ over the solid region in the first octant ($x \geq 0, y \geq 0, z \geq 0$) bounded above by $z = 8 - 2x^2 - y^2$ and below by $z = y^2$.

W]

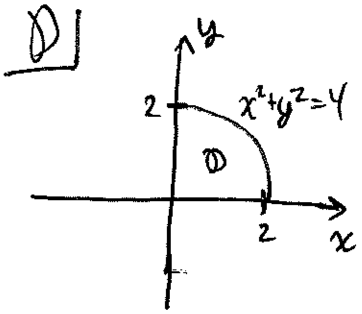


NEED INTERSECTION TO HELP FIND D

$$8-2x^2-y^2 = y^2 \rightsquigarrow 4 = x^2+y^2$$

SO WHEN WE COLLAPSE THE REGION DOWN TO xy-PLANE, WE SEE CIRCLE $x^2+y^2=4$ ... ONLY WANT FIRST QUADRANT:

D]



$$\text{SO} \quad \iiint_W f\,dV = \iint_D \int_{z=y^2}^{z=8-2x^2-y^2} x\,dz\,dA$$

$$= \iint_D xz\Big|_{z=y^2}^{z=8-2x^2-y^2} dA$$

$$= \iint_D \left(8x - 2x^3 - 2xy^2\right) dA$$

SWITCH TO POLARS FOR D: (AND TAKE OUT COMMON FACTOR OF x)

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2} \Big(8 \underbrace{-2r^2\cos^2\theta - 2r^2\sin^2\theta}_{-2r^2}\Big)(r\cos\theta)\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2} \left(8r^2\cos\theta - 2r^4\cos\theta\right) dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\frac{8}{3}\cdot 8\cos\theta - \frac{2}{5}\cdot 32\cos\theta\right) d\theta$$

$$= \frac{64}{3} - \frac{64}{5} = \frac{320-192}{15} = \frac{128}{15}$$

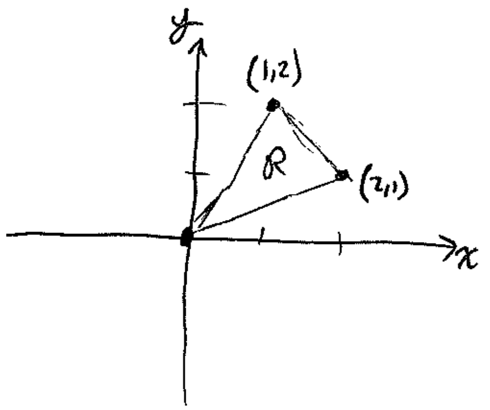
$$\begin{array}{r} 320 \\ -\,192 \\ \hline 128 \end{array}$$

2. [10 pts] Use the transformation $x = 2u + v$, $y = u + 2v$ to evaluate $\int\int_R (x - 3y)dA$ where R is the triangular region with vertices $(0,0), (2,1), (1,2)$ in the xy-plane. Be sure to draw both the original region R AND the resulting region in the uv-plane.

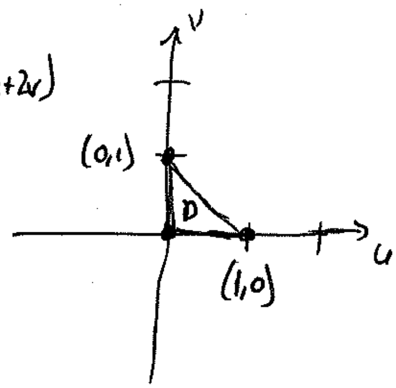
$G(0,0) = (0,0)$

Note $G(1,0) = (2,1)$

$G(0,1) = (1,2)$



$G(u,v) = (2u+v, u+2v)$



$$dG = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$|dG| = 3$$

$$\text{So } \iint_R (x-3y)\,dA = \iint_D (2u+v - 3(u+2v))\,|dG|\,dA$$

$$= -3 \int_{u=0}^{1}\int_{v=0}^{v=u} (+u + 5v)\,dv\,du$$

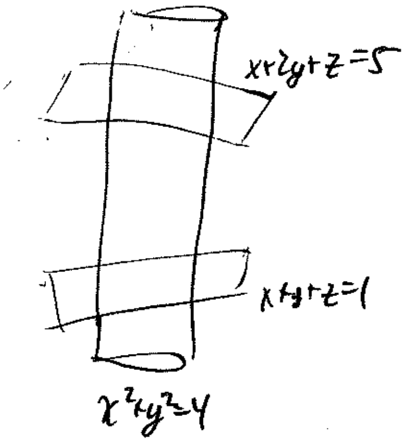
$$= -3 \int_{u=0}^{1} \left(uv + \frac{5}{2}v^2\right)\Big|_{v=0}^{u} du$$

$$= -3 \int_{u=0}^{1} u^2 + \frac{5}{2}u^2 \, du$$

$$= -\frac{21}{2}\int_{u=0}^{1} u^2\,du$$

$$= -\frac{7}{2}$$

3. [8 pts] Find the volume enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the plane $x + 2y + z = 5$, and bounded below by the plane $x + y + z = -1$.



$$\text{Volume} = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=1-x-y}^{z=5-x-2y} r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r(4-y)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (4r - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(8 - \frac{8}{3}\sin\theta\right) d\theta = 16\pi$$

4. [8 pts] Prove that div curl $\vec{\mathbf{F}}(x, y, z) = 0$ for any vector field $\vec{\mathbf{F}}$.

Let $\vec{F} = \langle F_1, F_2, F_3 \rangle$

Then $\operatorname{curl}(\vec{F}) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

So $\operatorname{div}\operatorname{curl}\vec{F} = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$

$= 0$

5. Suppose $f(x, y, z)$ is a scalar function, and $\overrightarrow{\mathbf{F}}(x, y, z)$ is a vector field (these two may be totally unrelated). Suppose we have a closed curve $\mathcal{C}$ in $\mathbb{R}^3$ and we know that $\oint_{\mathcal{C}} f\,ds = 5$ and $\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = 6$.

   (a) [3 pts] What is the value of $\oint_{-\mathcal{C}} f\,ds$? No justification needed.

   5

   (b) [3 pts] What is the value of $\oint_{-\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}}$? No justification needed.

   -6

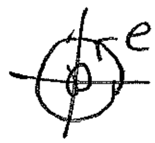
   (c) [4 pts] Is it possible that $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla} f$? Explain.

   NO. ~~[illegible] BUT~~ IF $\vec{F} = \vec{\nabla} f$, THEN WE MUST HAVE $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ SINCE $\mathcal{C}$ IS CLOSED. ~~BUT~~ BUT WE KNOW $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 6$.

6. [6 pts] Let C be the unit circle in $\mathbb{R}^2$, oriented counter-clockwise. Find $\oint_C \langle 2x - y, 2y + x\rangle \cdot d\vec{\mathbf{r}}$.

IF $\vec{F} = \langle 2x-y, 2y+x \rangle$ NOTE $\text{CURL}_z(\vec{F}) = 1 - (-1) = 2$

THUS $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{CURL}_z(\vec{F})\,dA = 2\ \text{AREA}(D) = 2\pi$



7. Let $f(x, y) = x^2 + 3y$, and let L be the line segment from $(3, 0)$ to $(0, 4)$.

(a) [5 pts] Find $\int_L f(x, y)ds$.

PARAMETRIZE L VIA $\vec{r}(t) = (1-t)\langle 3,0\rangle + t\langle 0,4\rangle = \langle 3-3t, 4t\rangle$

$\vec{r}\,'(t) = \langle -3, 4\rangle \qquad |\vec{r}\,'(t)| = 5$

SO $\int_L f\,ds = \int_{t=0}^{1}\left((3-3t)^2 + 3(4t)\right) \cdot 5\,dt = 5\int_{t=0}^{1}(9 - 12t + 9t^2 + 12t)\,dt$

$= 5(9+3) = 60$

(b) [5 pts] Find $\int_L \vec{\nabla} f(x, y) \cdot d\vec{\mathbf{r}}$

$\int_L \vec{\nabla} f \cdot d\vec{r} = f(\text{END}) - f(\text{START})$

$= f(0,4) - f(3,0)$

$= (0+12) - (9+0) = 3$

8. Let $\mathcal{S}$ be the cone given by equation $\phi = \frac{\pi}{6}$ in spherical coordinates.

(a) [4 pts] Find a parametrization $G(u, v)$ for $\mathcal{S}$.

LETTING ρ AND θ BE FREE: $u = \rho$, $v = \theta$

$x = \rho \sin\phi \cos\theta = \frac{1}{2}\rho\cos\theta$

$y = \rho \sin\phi \sin\theta = \frac{1}{2}\rho\sin\theta$

$z = \rho\cos\phi = \frac{\sqrt{3}}{2}\rho$

$$G(u,v) = \left(\frac{1}{2}u\cos v, \ \frac{1}{2}u\sin v, \ \frac{\sqrt{3}}{2}u\right)$$

(THERE ARE OTHER OPTIONS)

(b) [4 pts] Find the angle between the tangent vectors $\vec{\mathbf{T}}_u$ and $\vec{\mathbf{T}}_v$ at the point $(1, 0, \sqrt{3})$ (given in (x, y, z) coordinates).

AT POINT $(1,0,\sqrt{3})$, MUST HAVE $u=2$ AND THUS $v=0$

$$\vec{T}_u = \left\langle \frac{1}{2}\cos v, \ \frac{1}{2}\sin v, \ \frac{\sqrt{3}}{2}\right\rangle$$

$$\vec{T}_v = \left\langle -\frac{1}{2}u\sin v, \ \frac{1}{2}u\cos v, \ 0\right\rangle$$

AT $u=2, v=0$ THESE ARE $\vec{T}_u = \left\langle \frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right\rangle$

$\vec{T}_v = \langle 0, 1, 0\rangle$

ORTHOGONAL!

(THERE ARE OTHER OPTIONS)

(c) [6 pts] Find an equation for the tangent plane to the surface at that same point.

$$\vec{N} = \vec{T}_u \times \vec{T}_v = \left\langle -\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right\rangle$$

AT $u=2, v=0$

SO WE USE EQUATION $\left(\vec{PT} - \langle x, y, z\rangle\right) \cdot \vec{N} = 0$

$$\left(\langle 1, 0, \sqrt{3}\rangle - \langle x, y, z\rangle\right) \cdot \vec{N} = 0$$

$$-\frac{\sqrt{3}}{2}(1-x) + 0 + \frac{1}{2}(\sqrt{3} - z) = 0$$

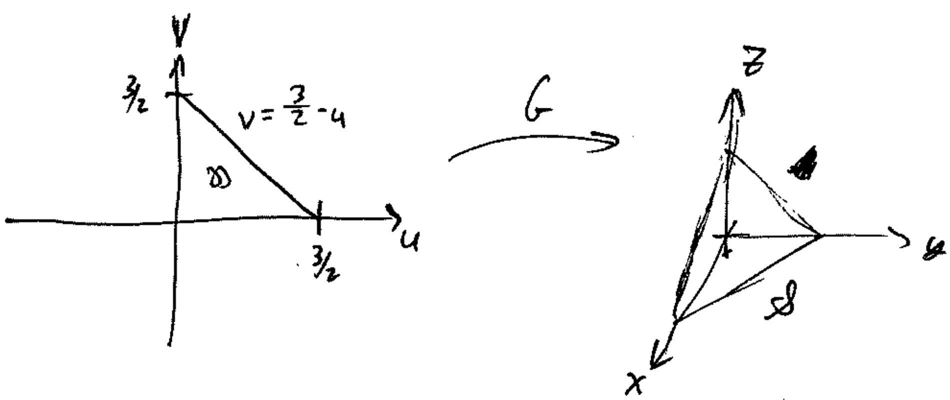
$$\frac{\sqrt{3}}{2}x - \frac{1}{2}z = 0$$

(OR ANY MULTIPLE OF THIS EQUATION)

9. [10 pts] The plane $x + 2y + 2z = 3$ has a finite portion $\mathcal{S}$ lying in the first octant $(x, y, z \geq 0)$. Suppose that this plane carries a charge density of $\delta_Q(x, y, z) = e^z$. Find the total charge Q on $\mathcal{S}$. Be sure to have a good sketch of your parametrization domain $\mathcal{D}$ if necessary.

WRITE AS $x = 3-2y-2z$, so $\begin{matrix} u=y \\ v=z \end{matrix} \rightsquigarrow G(u,v) = (3-2u-2v, u, v)$

NOW DOMAIN NEEDS
$x \geq 0 \rightsquigarrow 3-2u-2v \geq 0 \rightsquigarrow v \leq \frac{3}{2} - u$
$y \geq 0 \rightsquigarrow u \geq 0$
$z \geq 0 \rightsquigarrow v \geq 0$



$\vec{T}_u = \langle -2, 1, 0 \rangle$
$\vec{T}_v = \langle -2, 0, 1 \rangle$
$\vec{N} = \pm \langle 1, 2, 2 \rangle$
$|\vec{N}| = 3$

SO TOTAL CHARGE $Q = \iint_{\mathcal{S}} \delta_Q \, dS = \iint_{\mathcal{D}} \delta_Q(G(u,v)) \, |\vec{N}| \, dA$

$$= \iint_{\mathcal{D}} e^v \cdot 3 \, dA$$

$$= 3 \int_{u=0}^{3/2} \int_{v=0}^{3/2-u} e^v \, dv \, du$$

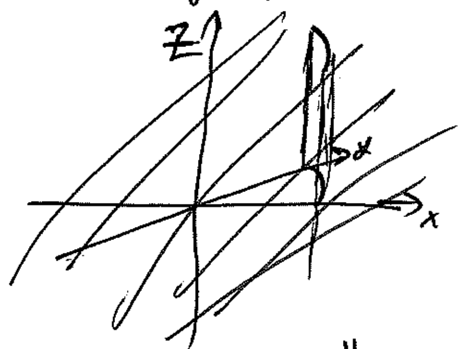
$$= 3 \int_{u=0}^{3/2} \left(e^{3/2-u} - 1 \right) du$$

$$= 3 \left[-e^{3/2-u} \Big|_{u=0}^{3/2} - u \Big|_{u=0}^{3/2} \right]$$

$$= 3 \left[-1 + e^{3/2} - \frac{3}{2} + 0 \right] = 3e^{3/2} - \frac{5}{2}$$

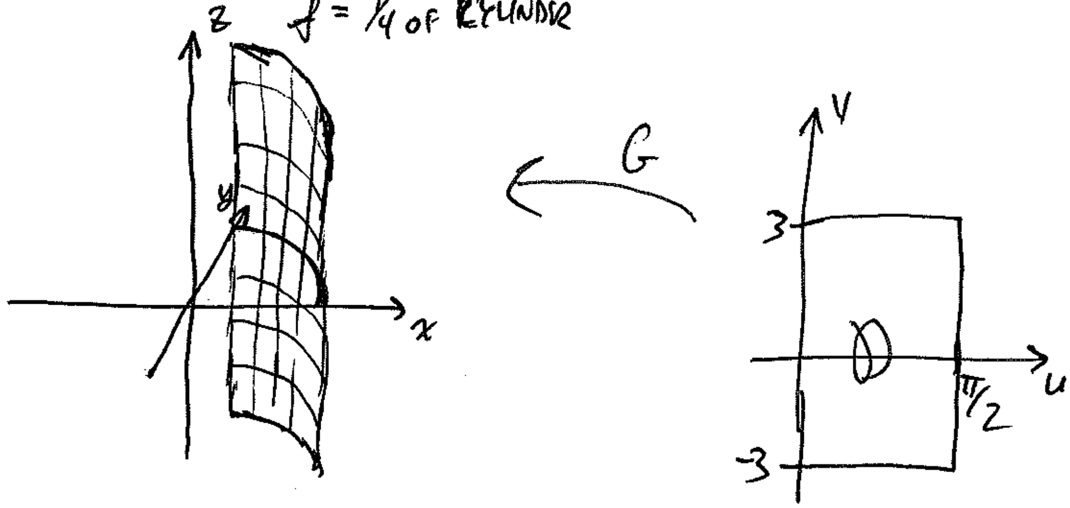
10. [10 pts] Compute the surface integral (flux) $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ of $\vec{\mathbf{F}} = \langle y, x, e^{xz} \rangle$ over the surface S defined by the equations $x^2 + y^2 = 9, x \geq 0, y \geq 0, -3 \leq z \leq 3$, with outward-pointing normal. Be sure to have a good sketch of your parametrization domain $\mathcal{D}$ if necessary.

$x^2+y^2=9$ IS CYLINDER OF RADIUS 3, AND WE ARE TRAPPED IN 1ST QUADRANT W/ Z GOING FROM −3 TO 3



USE CYLINDRICAL COORD'S TO PARAMETRIZE.
CLEARLY $r=3$ FIXED, WHILE $\theta: 0 \to \frac{\pi}{2}$
$z: 3 \to 3$

$\mathcal{S}$ = "1/4 OF CYLINDER"



$u=\theta$
$v=z$

$G(u,v) = (3\cos u, 3\sin u, v)$

$\vec{T}_u = \langle -3\sin u, 3\cos u, 0 \rangle$

$\vec{T}_v = \langle 0, 0, 1 \rangle$

$\vec{N} = \pm \langle 3\cos u, 3\sin u, 0 \rangle$

TO MAKE $\vec{N}$ <u>OUTWARDS</u> CHOOSE

$$\vec{N} = \langle 3\cos u, 3\sin u, 0 \rangle$$

$$\text{SO } \iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \int_{u=0}^{\pi/2} \int_{v=-3}^{3} \langle 3\sin u, 3\cos u, e^{3v\cos u} \rangle \cdot \langle 3\cos u, 3\sin u, 0 \rangle \, dA$$

$$= \int_{u=0}^{\pi/2} \int_{v=-3}^{3} 18 \sin u \cos u \, dv\, du$$

$$= 108 \int_{u=0}^{\pi/2} \sin u \cos u \, du$$

$$= 54 \sin^2 u \Big|_{u=0}^{\pi/2} = 54$$

11. [6 pts] Prove that, if $\vec{\mathbf{F}}$ is a conservative vector field, then the flux of Curl($\vec{\mathbf{F}}$) through any surface $\mathcal{S}$ (closed or not closed) must equal zero.

IF $\vec{F}$ IS CONSERVATIVE, THEN $\vec{F} = \vec{\nabla} f$ AND $\text{CURL}(\vec{F}) = \vec{0}$.

THUS $\iint_{\mathcal{S}} \text{CURL}\,\vec{F} \cdot d\vec{S} = 0$ REGARDLESS OF $\mathcal{S}$.

12. [6 pts] Find the volume of a region $\mathcal{W}$ if we know that

$$\iint_{\partial\mathcal{W}} \left\langle x + xy + z\,,\; x + 3y - \frac{1}{2}y^2\,,\; 4z \right\rangle \cdot d\vec{\mathbf{S}} = 16$$

BY DIVERGENCE THM, THIS INTEGRAL MATCHES

$$\iiint_{\omega} (1+y + 3-y + 4)\, dV$$

$$\parallel$$

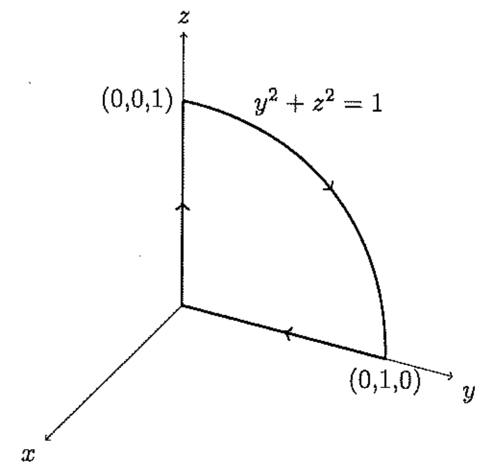
$$\iiint_{\omega} 8\, dV$$

$$\parallel$$

8 VOLUME(ω) THUS VOLUME$(\omega) = 2$

13. [6 pts] Evaluate $\oint_C \langle y, z, x\rangle \cdot d\vec{r}$ where C is the closed curve illustrated below (HINT: Stokes' Theorem).

$\vec{F}$



NOTE CURL $\vec{F} = \langle 1, -1, 1\rangle$ AND WE CAN FORM SURFACE $\mathcal{S}$ IN yz-PLANE, WITH UNIT NORMAL $\vec{n} = \langle -1, 0, 0\rangle$ TO MATCH ORIENTATION OF $\mathcal{C}$

SO $\int_{\mathcal{C}} \langle y, z, x\rangle \cdot d\vec{r} = \iint_{\mathcal{S}} \langle 1, -1, 1\rangle \cdot \vec{n}\, dS$

$= -\text{AREA}(\mathcal{S}) = -\frac{\pi}{4}$

14. TRUE/FALSE (circle your answer, no justification needed)

(a) [3 pts] Suppose that $\text{Curl}\,\vec{F}(x, y, z) = \vec{0}$ throughout some simply connected domain. Then there is one and only one potential function $f(x, y, z)$ such that $\vec{F} = \vec{\nabla} f$.

TRUE *FALSE*

(b) [3 pts] If $\vec{F}(x, y, z)$ is conservative on some domain (simply connected or not), then we must have $\text{Curl}\,\vec{F} = \vec{0}$.

TRUE *FALSE*