## Final Exam

Math 32B/2, Spring 2014

June 9

### Instructions:

- Read each question carefully before answering.
- There are eight equally-weighted problems.
- Carefully explain all of your work. In particular, cite any of the main theorems that you use.
- You may use one page of notes, as previously discussed. You MAY NOT use a calculator or any other electronic devices. Turn off and put away all mobile phones.

### Honor Statement:

*I hereby affirm that the work on this exam is my own, and that I have not consulted any unauthorized sources of information—including written, electronic, or human—during the exam.*

Name (signature): [redacted]

Name (printed): [redacted]

Student ID #: [redacted]

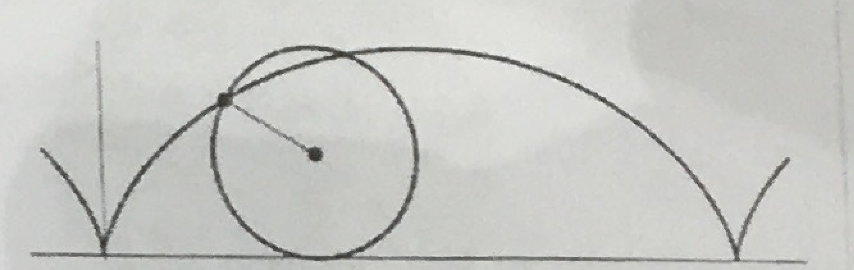
Section: A B C D E F

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points | 7 | 8 | 8 | 5 | 1 | 1 | 1 | 3 | 34 |

1. The *cycloid* is the curve traced by a designated point $P$ on a circle (of radius 1) as the circle rolls along the $x$-axis. If $P$ is at the origin at $t = 0$, then the cycloid can be parametrized by

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t).$$

Use Green's Theorem to find the area between one arc of the cycloid and the $x$-axis. [Hint: consider $\mathbf{F}(x, y) = \langle -y, 0\rangle$, and recall that $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$.]



Green's Theorem: $\oint_C F \cdot ds = \iint_D \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dA = \iint_D 1\, dA$

$F(x,y) = \langle -y, 0\rangle$

$c(t) = (t - \sin t, 1 - \cos t)$

$F(c(t)) = \langle \cos t - 1, 0\rangle$

$c'(t) = (1 - \cos t, \sin t)$

$F(c(t)) \cdot c'(t) = \langle \cos t - 1, 0\rangle \cdot \langle 1 - \cos t, \sin t\rangle$

$= 2\cos(t) - \cos^2(t) - 1$

$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)\, dt$

$\int_0^{2\pi} \left[2\cos(t) - \cos^2(t) - 1\right] dt$

$= \int_0^{2\pi} (2\cos t)\, dt - \int_0^{2\pi} (\cos^2 t)\, dt - \int_0^{2\pi} 1\, dt$

$= -\frac{1}{2}\int_0^{2\pi} (1 + \cos(2t))\, dt$

$u = 2t$

$\frac{1}{2} du = dt$

$= -\frac{1}{4}\int 1 + \cos(u)\, du$

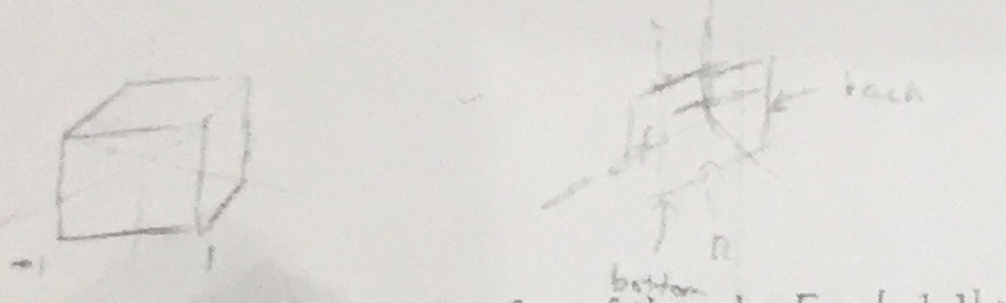
$= -\frac{1}{4}[2t + \sin(2t)]$

$\int_0^{2\pi} (2\cos t)\, dt - \int_0^{2\pi} \cos^2 t\, dt - \int_0^{2\pi} 1\, dt$

$= 2\sin t - \frac{t}{2} - \frac{1}{4}\sin(2t) - t \Big|_0^{2\pi}$

$= [0 - \pi - 2\pi] - [0]$

$= \boxed{-3\pi}$



2. Let $S$ be the outward-oriented surface of the cube $E = [-1,1] \times [-1,1] \times [-1,1]$ with the top face excluded, and let $\mathbf{F}(x,y,z) = \langle yz - x^3, 4y - z^2, 3x^2 z \rangle$. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$\iint_S F \cdot dS = \iint_{S\text{-front}} F \cdot dS + \iint_{S\text{-Back}} F \cdot dS + \iint_{S\text{-Left}} F \cdot dS + \iint_{S\text{-Right}} F \cdot dS + \iint_{S\text{-Bottom}} F \cdot dS$

$\iint_{S\text{-front}} F \cdot dS \to G(y,z) = (1, y, z) \to F(G) = \langle yz - 1, 4y - z^2, 3z \rangle$

$\vec{n} = (1,0,0)$ $\quad \vec{F}(G) \cdot \vec{n} = yz - 1$

$\int_{-1}^{1}\int_{-1}^{1} (yz - 1)\, dy\, dz = \int_{-1}^{1} \left(\frac{1}{2}y^2 z - y \Big|_{-1}^{1}\right) dz = \int_{-1}^{1} 0\, dz = 0$ $\quad$ X

$\iint_{S\text{-back}} F \cdot dS \to G(y,z) = (-1, y, z) \to F(G) = \langle yz + 1, 4y - z^2, 3z \rangle$

$\vec{n} = (-1,0,0)$ $\quad \vec{F}(G) \cdot \vec{n} = -yz - 1$

$\int_{-1}^{1}\int_{-1}^{1} (-yz - 1)\, dy\, dz = \int_{-1}^{1} \left(-\frac{1}{2}y^2 z - y \Big|_{-1}^{1}\right) dz = \int_{-1}^{1} -2\, dz = -4$ ✓

$\iint_{S\text{-left}} F \cdot dS \to G(x,z) = (x, -1, z) \to F(G) = \langle -z - x^3, -4 - z^2, 3x^2 z \rangle$

$\vec{n} = (0,-1,0)$ $\quad \vec{F}(G) \cdot \vec{n} = 4 + z^2$

$\int_{-1}^{1}\int_{-1}^{1} (4 + z^2)\, dx\, dz = \int_{-1}^{1} \left(4x + z^2 x \Big|_{-1}^{1}\right) dz = \int_{-1}^{1} 8 + 2z^2\, dz = 8z + \frac{2}{3}z^3 \Big|_{-1}^{1} = 16 + \frac{4}{3}$ ✓

$\iint_{S\text{-right}} F \cdot dS \to G(x,z) = (x, 1, z) \to F(G) = \langle z - x^3, 4 - z^2, 3x^2 z \rangle$

$\vec{n} = (0,1,0)$ $\quad \vec{F}(G) \cdot \vec{n} = 4 - z^2$

$\int_{-1}^{1}\int_{-1}^{1} (4 - z^2)\, dy\, dz = \int_{-1}^{1} 4y - z^2 y \Big|_{-1}^{1} dz = \int_{-1}^{1} (8 - 2z^2)\, dz = 8z - \frac{2}{3}z^3 \Big|_{-1}^{1} = 16 - \frac{4}{3}$ ✓

$\iint_{S\text{-bottom}} F \cdot dS \to G(x,y) = (x, y, -1) \to F(G) = \langle -y - x^3, 4y - 1, -3x^2 \rangle$

$\vec{n} = (0,0,-1)$ $\quad \vec{F}(G) \cdot \vec{n} = 3x^2$

$\int_{-1}^{1}\int_{-1}^{1} 3x^2\, dx\, dy = \int_{-1}^{1} x^3 \Big|_{-1}^{1} dy = \int_{-1}^{1} 2\, dy = 2y \Big|_{-1}^{1} = 4$ ✓

$\iint_S F \cdot dS = (-4) + (16 + \frac{4}{3}) + (16 - \frac{4}{3}) + (4)$

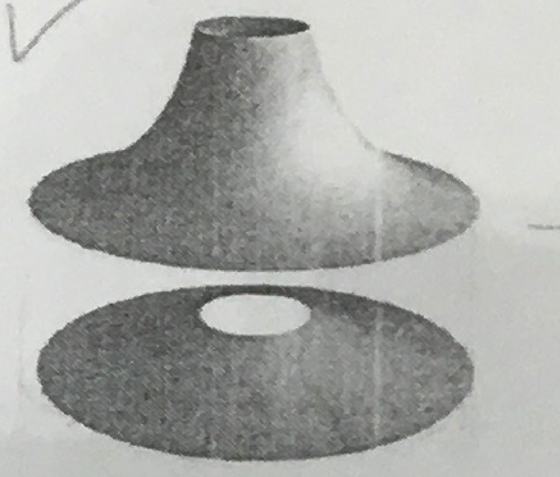
$$\boxed{\iint_S F \cdot dS = 32}$$

$-8$

3. Let $0 < a < b$ and define $E_{a,b} \subset \mathbb{R}^3$ as the region between the cylinders $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, and between the surfaces $z = \dfrac{1}{\sqrt{1 + x^2 + y^2}}$ and $z = \dfrac{1}{\sqrt{x^2 + y^2}}$, shown below.

8

(a) [8 points] Compute the volume of $E_{a,b}$ in terms of $a$ and $b$.



By Divergence Theorem

$\iiint_{E_{a,b}} dV = \text{Volume}(E_{a,b})$

$\frac{1}{\sqrt{1+r^2}} \le z \le \frac{1}{r}$

$x^2 + y^2 = b^2 \to r^2\cos^2\theta + r^2\sin^2\theta = b^2$

$r^2 = b^2 \to r = b$

$x^2 + y^2 = a^2 \to r = a; \quad a \le r \le b$

$0 \le \theta \le 2\pi$

$\iiint_{E_{a,b}} dV = \int_0^{2\pi}\int_{r=a}^{r=b}\int_{z=\frac{1}{\sqrt{1+r^2}}}^{z=\frac{1}{r}} r\, dz\, dr\, d\theta$

$= 2\pi \int_a^b r z \Big|_{\frac{1}{\sqrt{1+r^2}}}^{\frac{1}{r}} dr$

$= 2\pi \int_a^b \left(1 - \frac{r}{\sqrt{1+r^2}}\right) dr$

$= 2\pi \int_a^b 1\, dr - 2\pi \int_a^b \frac{r}{\sqrt{1+r^2}}\, dr$

$u = 1 + r^2 \to \frac{1}{2}du = r\, dr$

$\to 2\pi \int_a^b 1\, dr - 2\pi \int_a^b \frac{r}{\sqrt{1+r^2}}\, dr$

$= 2\pi\left[r - (1+r^2)^{1/2} \Big|_a^b\right]$

$= 2\pi\left[(b - (1+b^2)^{1/2}) - (a - (1-a^2)^{1/2})\right]$

$$= \boxed{2\pi\left[b - (1+b^2)^{1/2} - a + (1-a^2)^{1/2}\right]}$$

(b) [2 points] Show that $\lim_{a\to 0} \text{vol}(E_{a,1}) < \infty$ and $\lim_{b\to\infty} \text{vol}(E_{1,b}) < \infty$ (and so $\text{vol}(E_{0,\infty})$ is finite).

$\lim_{a\to 0} \text{vol}(E_{a,1}) = \lim_{a\to 0}\left[(1) - (1+1)^{1/2} - a + (1-a^2)^{1/2}\right]$ $\quad 2\pi$

$= \lim_{a\to 0}\left[1 - \sqrt{2} - a + (1-a^2)^{1/2}\right] \to \left[1 - \sqrt{2} - (0) + (1-(0)^2)^{1/2}\right] = (2 - \sqrt{2}) < \infty$

$\lim_{b\to 0} \text{vol}(E_{1,b}) = \lim_{b\to 0}\left[b - (1+b^2)^{1/2} - (1) + (1-(1)^2)^{1/2}\right]$

$-2 < 0$ ??

$= \lim_{b\to 0}\left[b - (1+b^2)^{1/2} - 1\right] \to \left[(0) - (1+(0)^2)^{1/2} - 1\right] = -2 < \infty$

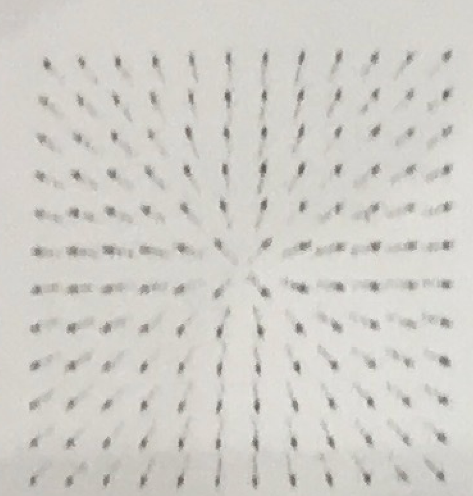
5

4. Let $\mathbf{F}$ be a vector field defined on a region $D \subset \mathbb{R}^2$. A curve $C$ is called a *flow curve* of $\mathbf{F}$ if it has a parametrization $\mathbf{c}(t)$ such that

$$\mathbf{F}(\mathbf{c}(t)) = \mathbf{c}'(t) \qquad (1)$$

for all $t$ where $\mathbf{c}(t)$ is defined. This means that at each point on $C$, $\mathbf{F}$ is a tangent vector for $C$, so the trajectory of the curve is entirely determined by the vector field.

(a) [5 points] Show that straight lines emanating from the origin are flow curves of the unit radial field $\mathbf{e}_r = \dfrac{(x, y)}{\sqrt{x^2 + y^2}}$. That is, find some $\mathbf{c}(t)$ so that equation (1) is true.



$e_r = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$  magnitude $= \sqrt{x^2 + y^2}$

$e_r = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$

$\rightarrow \left\langle \dfrac{\cos\theta}{\sqrt{\cos^2\theta + \sin^2\theta}}, \dfrac{\sin\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} \right\rangle$

$= \left\langle \dfrac{\cos\theta}{1}, \dfrac{\sin\theta}{1} \right\rangle = \langle \cos\theta, \sin\theta \rangle$

$\boxed{c(t) = \langle \cos t, \sin t \rangle}$

(b) [5 points] If $C$ is a flow curve of a vector field $\mathbf{F}$ (with orientation implied from equation (1)), show that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} \geq 0$.

Vector Line Integrals

$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)\, dt$

$= \int_a^b c'(t) \cdot c'(t)\, dt = \int_a^b \|c'(t)\|^2\, dt \geq 0$

$\int_C F \cdot ds \geq 0$

5. Let $S_R \subset \mathbb{R}^3$ be the sphere of radius $R > 0$, centered at the origin. Let $C$ be a simple closed curve on $S_R$ (an example is shown below) and consider the vector field $\mathbf{F}(x, y, z) = (x, y, x^2 + y^2 + z^2)$. Show that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for every such curve $C$. [Be sure to specify any orientations that you use.]

$F(x, y, z) = \langle x, y, x^2 + y^2 + z^2 \rangle$

$F = \nabla V$

$V_x = x$

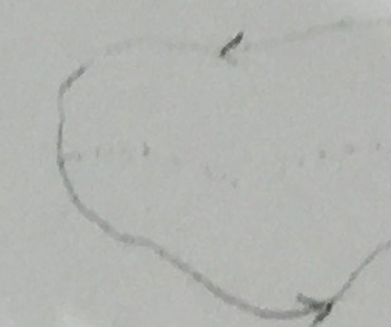
$V_y = y$

$V_z = x^2 + y^2 + z^2$

$V = \int x\, dx = \frac{1}{2}x^2 + c(y, z)$

$V = \int y\, dy = \frac{1}{2}y^2 + c(x, z)$

$V = \int (x^2 + y^2 + z^2)\, dz = x^2 z + y^2 z + \frac{1}{3}z^3 + c(x, y)$



$V = \frac{1}{2}x^2 + \frac{1}{2}y^2 + x^2 z + y^2 z + \frac{1}{3}z^3 + c(y, z) + c(x, z) + c(x, y)$

Since $C$ is a simple, closed curve and $F$ is a ~~conservative~~ vector field, $\oint_C F \cdot ds = 0$

6. Let $C$ be a simple curve in the $(y, z)$-plane with parametrization $c(t) = (y(t), z(t))$ for $a \le t \le b$. Assume that $y(t) > 0$ for all $t$. Let $S$ be the surface obtained by revolving $C$ around the $z$-axis. This has parametrization

$$G(t, \theta) = \big(y(t)\cos\theta, y(t)\sin\theta, z(t)\big), \qquad a \le t \le b, 0 \le \theta \le 2\pi.$$

An example of such a surface is shown below. Show that the surface area of $S$ equals $2\pi \int_C y\, ds$.

$G(t,\theta) = (y(t)\cos\theta,\ y(t)\sin\theta,\ z(t))$

Area $(S) = \iint f(x,y,z)\, dS$ only when $f=1$ !

$G(t,\theta)$ is parametrization using cylindrical coordinates

$r = y(t) \Rightarrow \iint r\, dr\, d\theta$ = Area

$\iint (y)\, dt\, d\theta = \int\int_0^{2\pi} y\, d\theta\, dt$

$= 2\pi \int_C y\, ds$

8. Let $E$ be a region in the $(u, v)$-plane with centroid $(\overline{u}, \overline{v})$. Let $G$ be a linear transformation from the $(u, v)$-plane into the $(x, y)$-plane with non-zero Jacobian. That is, for some $a, b, c, d \in \mathbb{R}$,

$$G(u, v) = (au + cv, bu + dv) = (x, y), \qquad \mathrm{Jac}(G)(u, v) = ad - bc \neq 0.$$

Let $(\overline{x}, \overline{y})$ be the centroid of the image $D = G(E)$. Show that $(\overline{x}, \overline{y}) = G(\overline{u}, \overline{v})$.
[Hint: Recall how area changes under linear transformations. Then compute $\overline{x}$ and $\overline{y}$ directly, using the definition of centroid and the Change of Variables Formula.]

$\overline{x} = \frac{1}{A} \iint_D x\, dA$

$\overline{y} = \frac{1}{A} \iint_D y\, dA$

Area $(G(D)) = |\mathrm{Jac}(G)|$ Area $(D)$

Area $(D) = \frac{\text{Area}(G(D))}{\mathrm{Jac}(G)}$

$\frac{1}{A} = \frac{ad-bc}{\text{Area}(G(D))}$

$\overline{x} = \frac{ad-bc}{\text{Area}(G(D))} \iint_E (au+cv)\, dA$

$\overline{y} = \frac{ad-bc}{\text{Area}(G(D))} \iint_E (bu+dv)\, dA$

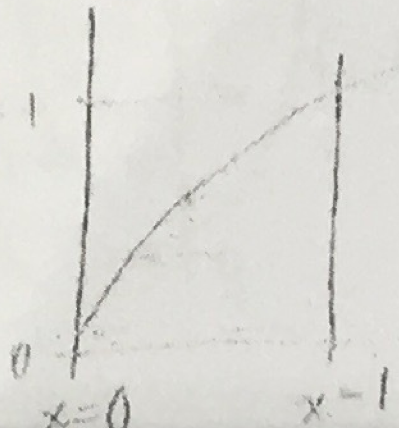
— 7

1. (10 pts) Compute the double integral

$$\int_0^1 \int_{y^2}^1 y^3 e^{x^3}\, dx\, dy$$

Can't find an antiderivative of $e^{x^3}$ with respect to $x$.
So we reverse the order of integration:

$$\begin{cases} y^2 \le x \le 1 \\ 0 \le y \le 1 \end{cases} \iff \begin{cases} 0 \le y \le \sqrt{x} \\ 0 \le x \le 1 \end{cases}$$



So $\displaystyle\int_{y=0}^{1}\int_{x=y^2}^{1} y^3 e^{x^3}\, dx\, dy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}} y^3 e^{x^3}\, dy\, dx$

$= \displaystyle\int_{x=0}^{1}\left(\frac{1}{4}y^4 e^{x^3}\Big|_{y=0}^{y=\sqrt{x}}\right) dx$

$\displaystyle\int_{x=0}^{1} \frac{1}{4}x^2 e^{x^3}\, dx$ $\qquad u = x^3$, $du = 3x^2\, dx$

$\displaystyle\frac{1}{12}\int_{x=0}^{1} 3x^2 e^{x^3}\, dx$

$= \displaystyle\frac{1}{12} e^{x^3}\Big|_0^1 = \frac{1}{12}(e^1 - e^0) = \boxed{\frac{e-1}{12}}$

2. (10 pts) Compute the following. (*Hint: Sketch the region and convert the whole thing to one integral.*)

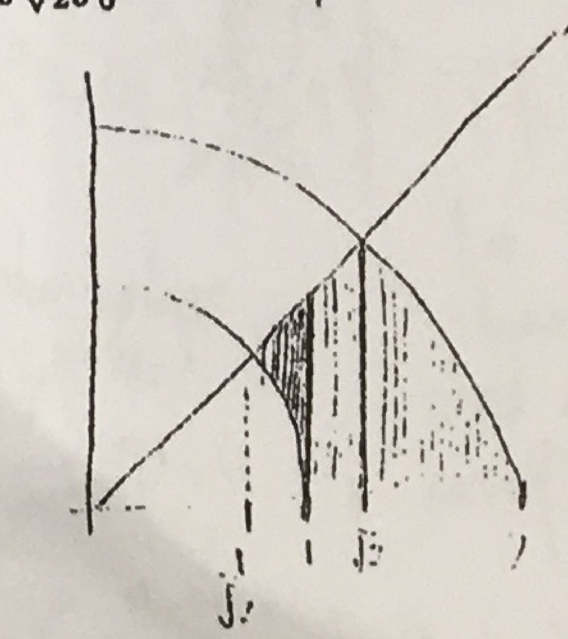
$y = \sqrt{4-x^2}$ $\quad x^2+y^2=4$

$$\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_1^{\sqrt{2}}\int_0^{x} xy\, dy\, dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\, dy\, dx$$

$y = \sqrt{1-x^2}$

$\Downarrow$

$x^2+y^2=1$



In polar coordinates, this region is easily described by

$$\begin{cases} 1 \le r \le 2 \\ 0 \le \theta \le \frac{\pi}{4} \end{cases}$$

$$= \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=1}^{2} (r\cos\theta)(r\sin\theta)\cdot r\, dr\, d\theta = \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=1}^{2} r^3 \sin\theta\cos\theta\, dr\, d\theta$$

$\displaystyle\int_{\theta=0}^{\frac{\pi}{4}} \sin\theta\cos\theta\, d\theta \cdot \int_{r=1}^{2} r^3\, dr$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

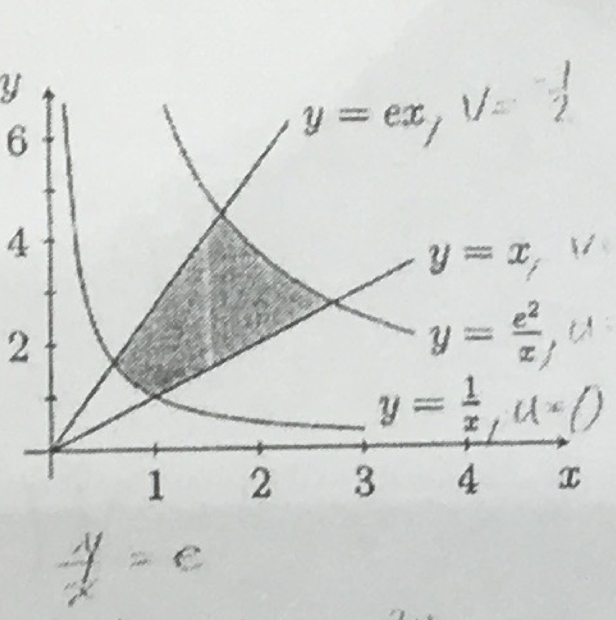
$= \left(\frac{1}{2}\sin^2\theta\Big|_{\theta=0}^{\frac{\pi}{4}}\right)\cdot\left(\frac{1}{4}r^4\Big|_{r=1}^{2}\right)$

$= \left(\frac{1}{2}\cdot\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2}(0)^2\right)\cdot\left(\frac{1}{4}\cdot 2^4 - \frac{1}{4}\cdot 1^4\right)$

$= \frac{1}{4}\cdot\left(\frac{16}{4} - \frac{1}{4}\right) = \boxed{\frac{15}{16}}$.

3. (10 pts) Let $\mathcal{D}$ be the region in the first quadrant bounded by $y = x$, $y = ex$, $y = \frac{1}{x}$, and $y = \frac{e^2}{x}$. Use the change of variables $x = e^{u+v}$, $y = e^{u-v}$ to compute the integral

$$\iint_{\mathcal{D}} \ln \frac{y}{x}\, dx\, dy.$$




$x = e^{u+v}$, $y = e^{u-v}$

$y = x \iff e^{u+v} = e^{u-v}$ or $\frac{y}{x} = 1$

$u+v = u-v$  $\frac{e^{u-v}}{e^{u+v}} = 1$

$2v = 0$  $= e^{-2v} = e^0$

$(v = 0)$

$y = ex \iff e^{u-v} = e \cdot e^{u+v} = e^{u+v+1}$ or $\frac{y}{x} = e$

$u - v = u + v + 1$  $\frac{e^{u-v}}{e^{u+v}} = e^1 = e^{-2v}$

$2v = 1$  $-2v = 1 \Rightarrow v = -\frac{1}{2}$

$(v = \frac{1}{2})$

$y = \frac{1}{x} \iff e^{u-v} = \frac{1}{e^{u+v}} = e^{-u-v}$ or $xy = 1$

$u - v = -u - v$  $e^{u+v} e^{u-v} = 1$

$2u = 0$  $e^{2u} = 1$

$(u = 0)$  $2u = 0 \Rightarrow u = 0$

$y = \frac{e^2}{x} \iff e^{u-v} = \frac{e^2}{e^{u+v}} = e^{2-u-v}$ or $xy = e^2$

$u - v = 2 - u - v$  $e^{u+v} e^{u-v} = e^2$

$2u = 2 \Rightarrow (u = 1)$  $e^{2u} = e^2$

$u = 1$

Jacobian: $\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} e^{u+v} & e^{u+v} \\ e^{u-v} & -e^{u-v} \end{bmatrix} = -e^{u+v} e^{u-v} - e^{u+v} e^{u-v} = -2e^{2u}$

So $|\det(J_{\phi})| = 2e^{2u}$

And $\ln\left(\frac{y}{x}\right) = \ln \frac{e^{u-v}}{e^{u+v}} = \ln(e^{-2v}) = -2v$.

So $\iint_{\mathcal{D}} \ln\left(\frac{y}{x}\right) dx\, dy = \int_{u=0}^{1} \int_{v=-\frac{1}{2}}^{0} -2v \cdot 2e^{2u}\, dv\, du = -4 \int_{u=0}^{1} e^{2u} du \cdot \int_{v=-\frac{1}{2}}^{0} v\, dv$

$= -4 \left[\frac{1}{2} e^{2u}\right]_{u=0}^{1} \left[\frac{1}{2} v^2\right]_{v=-\frac{1}{2}}^{0} = -(e^2 - 1)\left(0 - \frac{1}{4}\right) = \boxed{\frac{e^2 - 1}{4}}$

4. (10 pts) Let $\mathcal{S}$ be the surface $x^2 + y^2 - z^2 = -1$, where $1 \le z \le 3$. Suppose that $\mathcal{S}$ has mass density (per unit area) equal to $z$ at each point. What is the total mass of the surface $\mathcal{S}$?

$x^2 + y^2 = z^2 - 1$  $z = 1 \Longrightarrow r^2 = 0$  $z = 3 \Longrightarrow r^2 = 9 - 1 = 8$

In cylindrical: $r^2 = z^2 - 1$  $(r = 0)$  $r = 2\sqrt{2}$

Cross sections (w/ constant $z$) are circles.

Parametrization: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{r^2 + 1} \end{cases}$  $0 \le \theta \le 2\pi$, $0 \le r \le 2\sqrt{2}$

$\vec{T}_r = \left\langle \cos\theta, \sin\theta, \frac{1}{2}(r^2+1)^{-1/2} \cdot 2r \right\rangle = \left\langle \cos\theta, \sin\theta, \frac{r}{\sqrt{r^2+1}} \right\rangle$

$\vec{T}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{n} = \vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \frac{r}{\sqrt{r^2+1}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left\langle \frac{-r^2\cos\theta}{\sqrt{r^2+1}}, \frac{-r^2\sin\theta}{\sqrt{r^2+1}}, r \right\rangle$

$= r\left\langle \frac{-r\cos\theta}{\sqrt{r^2+1}}, \frac{-r\sin\theta}{\sqrt{r^2+1}}, 1 \right\rangle$

$\|\vec{n}\| = \|\vec{T}_r \times \vec{T}_\theta\| = r\left(\frac{r^2\cos^2\theta + r^2\sin^2\theta}{r^2+1} + 1\right)^{1/2} = r\sqrt{\frac{r^2}{r^2+1} + 1} = r\sqrt{\frac{2r^2+1}{r^2+1}}$

Mass $= \iint_{\mathcal{S}} z\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{2\sqrt{2}} \sqrt{r^2+1}\; r\sqrt{\frac{2r^2+1}{r^2+1}}\, dr\, d\theta$

$= \int_{\theta=0}^{2\pi} 1\, d\theta \int_{r=0}^{2\sqrt{2}} r\sqrt{2r^2+1}\, dr = 2\pi \cdot \frac{1}{4} \int_{r=0}^{2\sqrt{2}} 4r\sqrt{2r^2+1}\, dr$

$u = 2r^2 + 1$

$du = 4r\, dr$

$= \frac{\pi}{2} \cdot \frac{2}{3}(2r^2+1)^{3/2}\Big|_{r=0}^{2\sqrt{2}} = \frac{\pi}{3}\left(17^{3/2} - 1^{3/2}\right) = \boxed{\frac{\pi}{3}\left(17\sqrt{17} - 1\right)}$

part of the problem ... know moments of inertia by ... you can still do the problem ... inertia that you need, you can still do ... ent of inertia I stand ... formulas (and state clearly what moment of inertia ... The system starts from rest. ... can slide without friction. ... before reaching the horizontal surface. ... blocks along the horizontal surface.

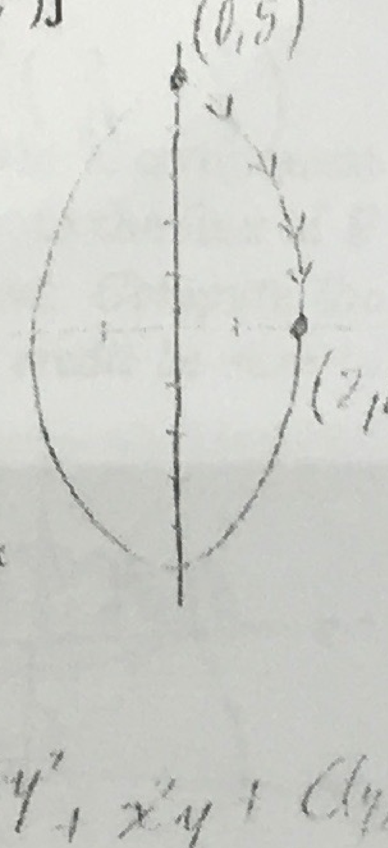
5. (10 pts) Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$, where

$$\mathbf{F}(x,y) = (xe^{x^2+y^2} + 2xy)\mathbf{i} + (ye^{x^2+y^2} + x^2)\mathbf{j}$$

and $C$ is the portion of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

in the first quadrant, oriented clockwise.




Check if F is conservative (optional):

$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$? $2xye^{x^2+y^2}+2x = 2xye^{x^2+y^2}+2x$ ✓

Can we find a potential function?

$f = \int F_1\,dx = \int (xe^{x^2+y^2} + 2xy)\,dx = \frac{1}{2}e^{x^2+y^2} + x^2y + C(y)$

$\frac{\partial f}{\partial y} = ye^{x^2+y^2} + x^2 + C'(y) = ye^{x^2+y^2} + x^2$

$C'(y) = 0$

$C(y) = \text{constant.}$

So $f(x,y) = \frac{1}{2}e^{x^2+y^2} + x^2y$ is a potential function for F.

So by the Fundamental Theorem of Line Integrals:

$\int_C \vec{F}\cdot d\vec{s} = f(\text{end point}) - f(\text{start point}) = f(2,0) - f(0,5)$

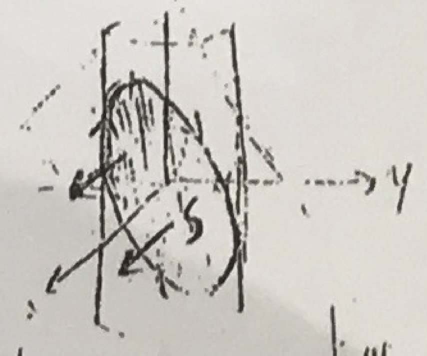
$= (\frac{1}{2}e^4 + 0) - (\frac{1}{2}e^{25} + 0) = \boxed{\frac{e^4 - e^{25}}{2}}$

This line integral can be computed directly, but it's easier to do it using Stokes' Theorem...

6. (10 pts) Let $C$ be the curve of intersection of the cylinder $x^2+y^2=9$ and the plane $x+y+z=0$, oriented clockwise as viewed from above. (This happens to be an ellipse.) Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$, where

$$\mathbf{F}(x,y,z) = \langle x^2z,\ y^2x,\ z^2y\rangle.$$

$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & y^2x & z^2y \end{vmatrix} = \langle z^2, x^2, y^2 \rangle$




For Stokes' Theorem, we need downward normal vectors!

Parametrization of the region of the plane $x+y+z=0$ inside the cylinder $x^2+y^2=9$:

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = -x-y = -r\cos\theta - r\sin\theta \end{cases}$ $\quad 0 \le r \le 3$, $0 \le \theta \le 2\pi$

$\vec{T}_r = \langle \cos\theta, \sin\theta, -\cos\theta - \sin\theta\rangle$

$\vec{T}_\theta = \langle -r\sin\theta, r\cos\theta, r\sin\theta - r\cos\theta\rangle$

$\vec{n} = \vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -\cos\theta-\sin\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta - r\cos\theta \end{vmatrix} = \langle r\sin^2\theta - r\sin\theta\cos\theta + r\cos^2\theta + r\sin\theta\cos\theta,\ r\sin\theta\cos\theta + r\sin^2\theta - r\sin\theta\cos\theta + r\cos^2\theta,\ r\cos^2\theta + r\sin^2\theta\rangle$

$= \langle r, r, r\rangle \leftarrow$ These point upward... so reverse this.

$\vec{n} = \vec{T}_\theta \times \vec{T}_r = \langle -r, -r, -r\rangle$

By Stokes' Theorem,

$\int_C \vec{F}\cdot d\vec{s} = \iint_S \langle z^2, x^2, y^2\rangle \cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} \langle (r\cos\theta + r\sin\theta)^2, r^2\cos^2\theta, r^2\sin^2\theta\rangle \cdot \langle -r, -r, -r\rangle\,dr\,d\theta$

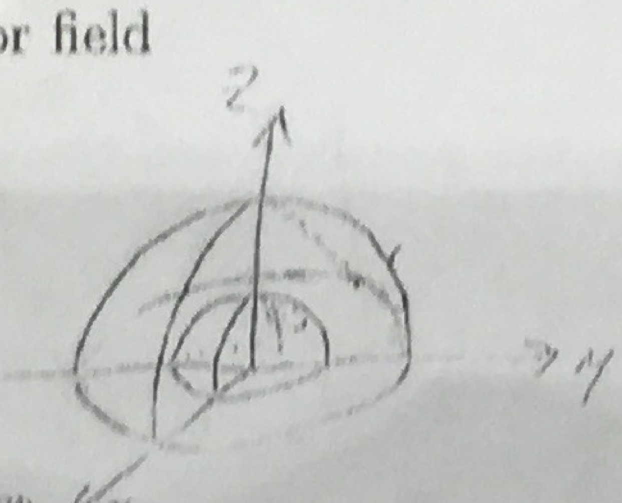
$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3} -r^3(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta + \sin^2\theta)\,dr\,d\theta$

$= -\int_{r=0}^{3} r^3\,dr \cdot \int_{\theta=0}^{2\pi} (2 + 2\sin\theta\cos\theta)\,d\theta = -\left[\frac{1}{4}r^4\right]_{r=0}^{3} \cdot \left[2\theta + \sin^2\theta\right]_{\theta=0}^{2\pi}$

$= -(\frac{1}{4}\cdot 81 - \frac{1}{4}\cdot 0)\cdot(2\cdot 2\pi - 0) = \boxed{-81\pi}$

7. (10 pts) Let $\mathcal{W}$ be the region described by $1 \le x^2 + y^2 + z^2 \le 4$ and $z \ge 0$, and let $\mathcal{S}$ be the boundary surface of $\mathcal{W}$. Compute the flux out of $\mathcal{S}$ (i.e., with outward normal vectors) of the vector field

$$\mathbf{F}(x, y, z) = \langle x^3,\ y^3,\ z^3 \rangle.$$



$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\,dV$$

By Divergence Theorem

$$\operatorname{div}(\vec{F}) = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

$3\rho^2$ in spherical coordinates

Since $W$ is the region between two hemispheres, and $\operatorname{div}(\vec{F}) = 3\rho^2$, we'll use spherical coordinates: $1 \le \rho^2 \le 4$, so $\begin{cases} 1 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \end{cases}$

$$\text{So flux} = \iiint_W \operatorname{div}(\vec{F})\,dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=1}^{2} 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{\phi=0}^{\pi/2} \sin\phi\,d\phi \cdot \int_{\rho=1}^{2} 3\rho^4\,d\rho$$

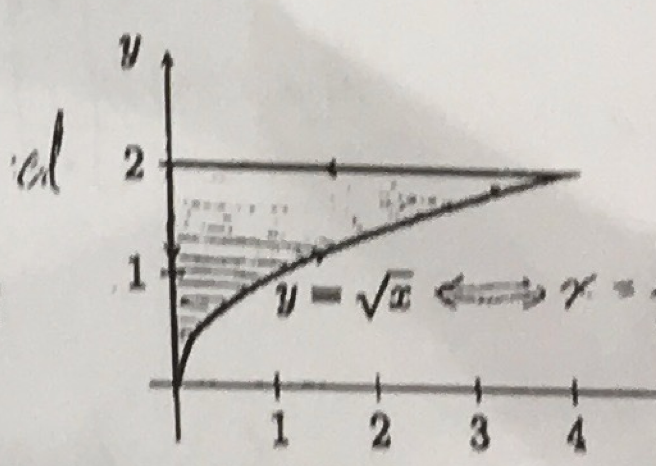
$$= 2\pi\cdot\left[-\cos\phi\right]_{\phi=0}^{\pi/2}\cdot\left[\tfrac{3}{5}\rho^5\right]_{\rho=1}^{2} = 2\pi\cdot(0-(-1))\cdot\left(\tfrac{96}{5} - \tfrac{3}{5}\right)$$

$$= \boxed{\frac{186\pi}{5}}$$

8. (10 pts) Let $C$ be the curve shown in the figure below. Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$ where

$$\mathbf{F}(x, y) = \langle \cancel{\ln x}^{\,\cos x} - xy,\ x + \sin(y^2) \rangle.$$

Oops!




To compute this directly, we would need to compute 3 separate line integrals, and the line integral along the vertical segment would involve an integral of $\sin(y^2)dy$, which can't be evaluated analytically!

But $C$ is a closed curve, so we can try Green's Theorem:

$$\int_C \vec{F}\cdot d\vec{s} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA = \int_{y=0}^{2}\int_{x=0}^{y^2}(1 - -x)\,dx\,dy \quad \left(\text{or } \int_{x=0}^{4}\int_{y=\sqrt{x}}^{2}(1+x)\,dy\,dx\right)$$

$$= \int_{y=0}^{2}\left[x + \tfrac{1}{2}x^2\right]_{x=0}^{y^2} dy$$

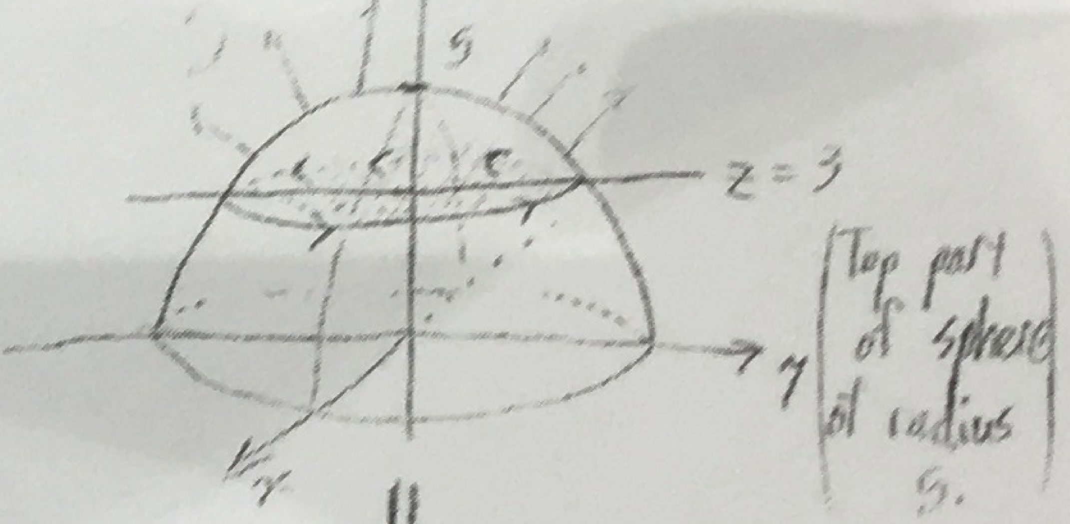
$$= \int_{y=0}^{2}\left(y^2 + \tfrac{1}{2}y^4\right)dy = \tfrac{1}{3}y^3 + \tfrac{1}{10}y^5\Big|_0^2 = \tfrac{8}{3} + \tfrac{32}{10} = \boxed{\frac{88}{15}}$$

9. (10 pts) Let $S$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ where $z \geq 3$. Suppose $\mathbf{F}$ is a vector field of the form

$$\mathbf{F}(x, y, z) = \langle F_1(x, y, z),\ F_2(x, y, z),\ z \rangle$$

(i.e., the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{F}$ are unspecified, and the $\mathbf{k}$ component is $z$), and assume that $\mathbf{F}$ has a vector potential. Compute the flux of $\mathbf{F}$ through $S$, oriented with upward normal vectors. *(Hint: Compute the integral over a different, but related, surface. For full credit be sure to justify all of your steps.)*

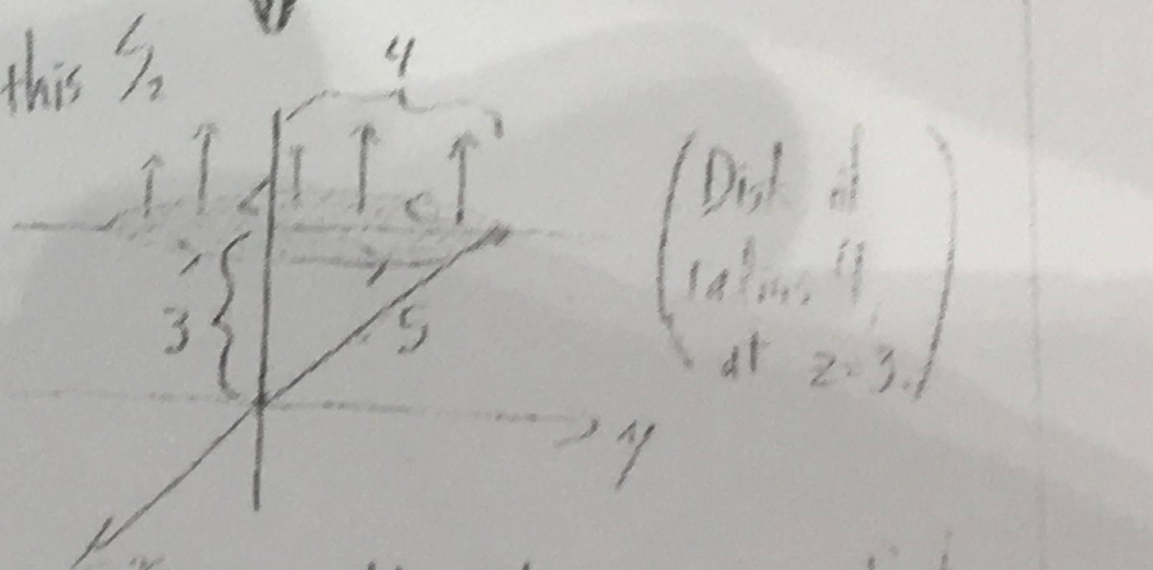
Since $\vec{F}$ has a vector potential (i.e. $\vec{F} = \text{curl}(\vec{A})$ for some vector field $\vec{A}$), the flux of $\vec{F}$ (surface integral of $\vec{F}$) is "independent of surface." Or in other words:




$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \text{curl}(\vec{A})\cdot d\vec{S} \overset{\text{by Stokes'}}{=} \oint_{\partial S} \vec{A}\cdot d\vec{s}$$

$$= \oint_{\partial S_2} \vec{A}\cdot d\vec{s} = \iint_{S_2} \text{curl}(\vec{A})\cdot d\vec{S}$$

Call this $S_2$




$$= \iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot\vec{e}_3\, dS$$

$$= \iint_{S_2} \vec{F}\cdot\vec{k}\, dS = \iint_{S_2} z\, dS$$

Normal vector is $\vec{k}$!

$$= \iint_{S_2} 3\, dS = 3\iint_{S_2} 1\, dS$$

$$3\cdot(\text{area of } S_2) = 3\cdot\pi\cdot 4^2 = \boxed{48\pi}$$