4.5

**Exercise 1** (6 points).

(1) Compute $\iiint_D (x+y)\,dV$, where $D := \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + y^2 + z^2 \le 4\}$.

*Hint:* you may use spherical coordinates.

(2) Find the total mass of a solid right circular cone with radius 3 and height 4, for which the density of each point of it is given by the distance from the base.

*Hint:* you may use cylindrical coordinates, and mind that the radius is always depending on the actual height.

1) let $u = x-1$, $v = y$, $w = z$

$D \to u^2+v^2+w^2 \le 4$

$x+y = u+v+1$

centers sphere at origin

$-2 \le u \le 2; -2 \le v \le 2; -2 \le w \le 2$

$Jac(G) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

$\iiint_{D} u+v+1 \, dV_{u,v,w}$

use spherical $H(r,\theta,\phi) = (r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi)$ $\quad Jac(H) = r^2\sin\phi$

$0 \le r \le 2;\ 0 \le \theta \le 2\pi;\ 0 \le \phi \le \pi$

$\int_0^2\int_0^{2\pi}\int_0^{\pi} (r\cos\theta\sin\phi + r\sin\theta\sin\phi + 1)\, r^2\sin\phi \; d\phi\, d\theta\, dr$

$\int_0^2\int_0^{2\pi}\int_0^{\pi} r^3\cos\theta\sin^2\phi + r^3\sin\theta\sin^2\phi + r^2\sin\phi \; d\phi\, d\theta\, dr$

+1.5

2) $\delta(x,y,z) = z$

$\delta(r,\theta,z) = z$

$0 \le z \le 4$ $\quad$ cylindrical: $G(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

$0 \le r \le \frac{3}{4}(4-z)$ $\quad$ $Jac(G) = r$

$M = \int_0^{2\pi}\int_0^{4}\int_0^{\frac{3}{4}(4-z)} z r \, dr\, dz\, d\theta = \frac{9}{32}\int_0^{2\pi}\int_0^4 16z - 8z^2 + z^3 \, dz\, d\theta$

$= \frac{9}{32}\int_0^{2\pi} \left(8z^2 - \frac{8}{3}z^3 + \frac{z^4}{4}\right)\Big|_0^4 d\theta$

$= \frac{9\pi}{16}\left(8z^2 - \frac{8}{3}z^3 + \frac{z^4}{4}\right)\Big|_0^4$

+3

$= \left(\frac{9\pi z^2}{2} - \frac{3\pi}{2}z^3 + \frac{9\pi z^4}{4^3}\right)\Big|_0^4$

$= 72\pi - 96\pi + 36\pi = \boxed{12\pi}$

**Exercise 2** (7 points).

Alice decided that she goes hiking. The path that she has chosen can be described analytically in $\mathbb{R}^2$ as follows. She is starting at the point $a = (-1, 0)$ and goes up on the upper half part of the unit circle (centered at the origin) until $(0, 1)$ then she goes down on the same circle until $b = (1, 0)$. From $b$ she then goes up on a straight line to the point $c = (2, 1)$. Then she continues straight to the point $d = (3, 1)$ and lastly she descends on the straight line to $e = (4, 0)$. We assume that on the whole road she is affected only by the gravitational field, which is given by the constant vector field $G = (0, -2)$.

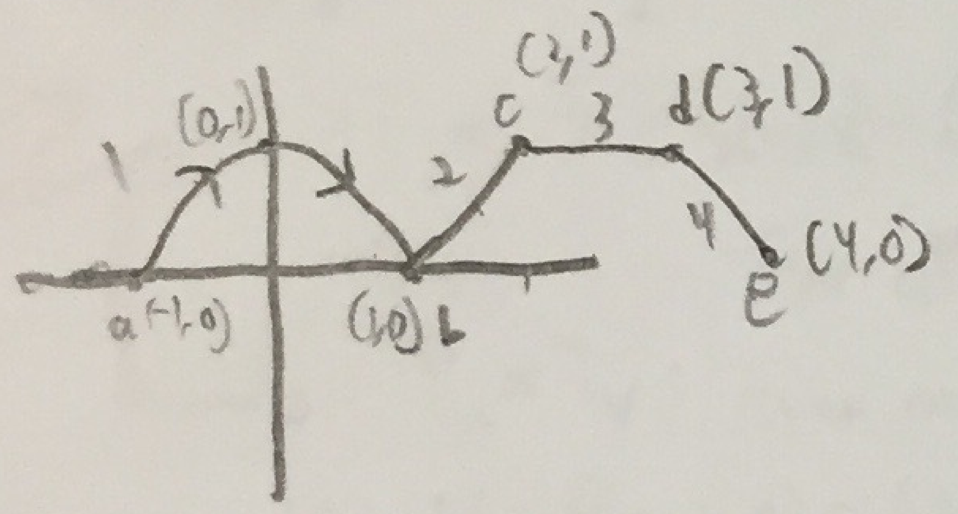
(1) Give a parametrization for each piece of path (you have 4 pieces: the circular, the straight up, the horizontal and the straight down ones) such that Alice has constant speed on each piece of path, and she needs $\pi$ time units for the circular piece, and 1 time unit on any other piece.

(2) Compute the total work she needs to perform in order to arrive from $a$ to $e$ on the path with the parametrization described in (1) and knowing that she is only affected by the gravitational field $G$.

*Hint:* you may use the additive property of line integrals with respect to concatenations of curves. You may use any other properties of the line integrals or the geometry of the problem in your solution.

(3) When she arrives to the point $e$, she meets Ben, who decided to hike on the very same path, but starting from $e$ and towards $a$ (using the opposite parametrization of the one in (1)). Knowing Alice's work and the fact that he is also affected only by the same gravitational field $G$, what is the total amount of work that Ben has to perform on this road? Justify your answer!

(4) After finishing the hike, Alice realizes that the gravitational field "helps her" always when she is going downwards. Would the amount of work she performed change if she doubled her speed on the path from $d$ to $e$? If yes, how does the total amount of work change in this case? Justify your answer!




$G = (0, -2)$

1) $r_1(t) = (-\cos t, \sin t) \quad 0 \le t \le \pi$ $\quad r_2(t) = (t+1, t) \quad 0 \le t \le 1$ $\quad r_3(t) = (t+2, 1) \quad 0 \le t \le 1$

$r_4(t) = (t+3, 1-t) \quad 0 \le t \le 1$

2) let $g(x,y) = -2y$. $\nabla g = (0, -2) = G$, so $G$ is conservative with potential function $g(x,y) = -2y$. Thus, $\int_C G \cdot dr$ is path-independent

$$W_{\text{against } G} = -\int_a^e G \cdot dr = -(g(e) - g(a)) = -((-2 \cdot 0) - (-2 \cdot 0)) = \boxed{0}$$

3) The field $G$ is still conservative and thus path-independent. However, Ben travels in the reverse direction, so $W_{Ben} = -W_{Alice}$ by the principle of reversing orientation. Therefore $W_{Ben} = \boxed{0}$

4) No. Since $G$ is a conservative field, the work performed by Alice is path-independent. This means it depends only on the value of the potential function at the endpoints, not the parameterization of the curve that gets her there. Changing her speed from $d$ to $e$ only changes how her path is parameterized, not her endpoint. Thus, changing her speed will not affect the total amount of work she does.

**Exercise 3** (7 points).

Let us consider the surface $K$ of the cube centered at the origin with side length 3 in $\mathbb{R}^3$; i.e. the surface of $\{(x, y, z) \in \mathbb{R}^3 : -3/2 \le x \le 3/2;\ -3/2 \le y \le 3/2;\ -3/2 \le z \le 3/2\}$.

2/2 (1) Compute the *unit normal vectors* $N(P)$ of $K$ pointing outward, where $P$ is an arbitrary point varying on the 6 facets of the cube.

*Hint:* discuss 6 cases. You do not have to discuss the cases when $P$ is on the edges or on the vertices of $K$.

Now let us consider the intersection of $K$ with the $xy-$plane, i.e. $C = K \cap \{z = 0\}$ which defines a closed piecewise linear curve, given by the edges of the planar square centered at the origin with side length 3. We orient $C$ clockwise.

0/1 (2) Compute $\int_C N \cdot dr$! Justify your answer!

1.5/2 (3) Let us consider the following vector field defined as $F(x, y) = \left(\frac{1}{(2-x)^2(2-y)}, \frac{1}{(2-x)(2-y)^2}\right)$. Show that $F$ is a conservative vector field on a domain of $\mathbb{R}^2$ that should be determined! Find a potential function of $F$ on its domain.

1.5/2 (4) Compute $\int_C F \cdot dr$ and $\int_{\tilde{C}} F \cdot dr$, where $C$ was defined previously and $\tilde{C}$ is the line segment joining the point $(-3/2, -3/2)$ to $(3/2, 3/2)$ (oriented from $(-3/2, -3/2)$ towards $(3/2, 3/2)$) i.e. a diagonal of $K$.

Parameterizations

1) Case 1: $x = -\frac{3}{2}$: $G_1(u,v) = (-\frac{3}{2}, u, v)$ 3: $y = -\frac{3}{2}$: $(u, -\frac{3}{2}, v) = G_3(u,v)$ 5: $z = -\frac{3}{2}$: $(u, v, -\frac{3}{2}) = G_5(u,v)$

2p 2: $x = \frac{3}{2}$: $G_2(u,v) = (\frac{3}{2}, u, v)$ 4: $y = \frac{3}{2}$: $(u, \frac{3}{2}, v) = G_4(u,v)$ 6: $z = \frac{3}{2}$: $(u, v, \frac{3}{2}) = G_6(u,v)$

For all cases, $-\frac{3}{2} \le u \le \frac{3}{2}$, $-\frac{3}{2} \le v \le \frac{3}{2}$

Normals: $T_u \times T_v$: even cases are $-$ of odd cases since parallel, but opp. direction to point outwards

2) 1: $N_1 = -[(0,1,0) \times (0,0,1)] = \langle -1, 0, 0 \rangle$ ✓ 3: $N_3 = -[(1,0,0) \times (0,0,1)] = \langle 0, -1, 0 \rangle$ ✓ 5: $N_5 = -[(1,0,0) \times (0,1,0)] = \langle 0, 0, -1 \rangle$ ✓

2: $N_2 = -N_1 = \langle 1, 0, 0 \rangle$ ✓ 4: $N_4 = -N_3 = \langle 0, 1, 0 \rangle$ ✓ 6: $N_6 = -N_5 = \langle 0, 0, 1 \rangle$ ✓

0p 2) $\int_C N \cdot dr = \int_{C_1} N_1 \cdot dr + \int_{C_2} N_2 \cdot dr + \int_{C_3} N_3 \cdot dr + \int_{C_4} N_4 \cdot dr$

Each of these is equal since the normal w.r.t. its path is the same for all sides let $r(t) = (t, 0, 0)$ $N(r(t)) = (1,0,0)$ $N \cdot r' = 1$

NO, the normal is always $\perp$ to C!

$\int_C N \cdot dr = 4 \int_{-\frac{3}{2}}^{\frac{3}{2}} 1\, dt = 4(t)\Big|_{-\frac{3}{2}}^{\frac{3}{2}} = \boxed{12}$ $= 0$

4) on back

1.5p

not just this point. The 2 lines $x=2$ and $y=2$!

3) Let $D: \mathbb{R}^2 \setminus \{(2,2)\}$, then $F$ is cons. if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

$-\frac{1}{(2-x)^2(2-y)^2} = -\frac{1}{(2-x)^2(2-y)^2}$ ✓

so $F$ is cons. on $D$

$\int F_1\, dx = \int F_2\, dy$

$f$: $\frac{-1}{(2-x)(2-y)} + g(y) = -\frac{1}{(2-x)(2-y)} + h(x)$

let $g(y)$ and $h(x) = 0$

$-\frac{1}{(2-x)(2-y)} = -\frac{1}{(2-x)(2-y)}$

let $\boxed{f = -\frac{1}{(2-x)(2-y)}}$ $\nabla f = F$ so $f$ is pot. function on $D$

4) Neither $C$ nor $\tilde{C}$ approaches $(2,3)$ where $F$ is not conservative. not just this one is bad!

1.5p Thus, since $C \in D$ and $\tilde{C} \in D$, $F$ can be treated as cons.

$$\int_C F \cdot dr = \boxed{0}$$

Since $C$ is closed path and $F$ is cons.

$$\int_{\tilde{C}} F \cdot dr = f\left(-\tfrac{3}{2}, -\tfrac{3}{2}\right) - f\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$

$$= \frac{-1}{C^{14}}$$

?