2/7

Exercise 1 (7 points).

Let D be the following two dimensional domain: $D := \{(x,y) \in \mathbb{R}^2 : \ 1 \le x \le e; 0 \le y \le \ln(x)\}$, where ln denotes the natural logarithm and the number e denotes its base, i.e. $e \approx 2.71$.

(1) Sketch the domain D and compute its area.

(2) Let us define the domain $\widetilde{D}$ as the part of the rectangle $[1,e] \times [0,1]$ that is above D. Sketch $\widetilde{D}$ and compute its area.

(3) Compute $\displaystyle\int\int_D \frac{e^{2y}}{x}\, dA(x,y)$.

Now rotate the domain D around the x−axes to obtain a solid 3D object which is axially symmetric w.r.t. the x−axes. Call this domain W.

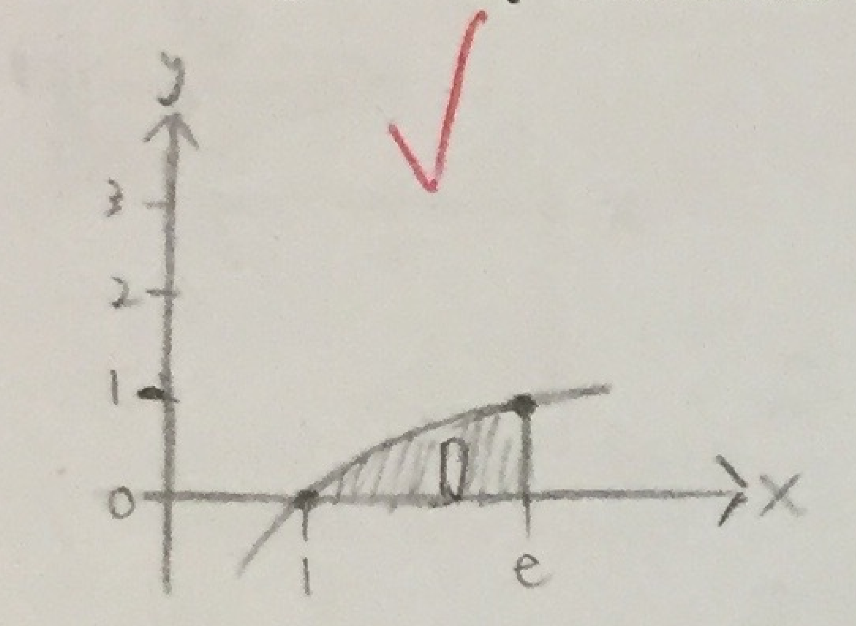
(4) Sketch the domain W and compute its volume.

Hint: find out what are the cross sections of this object if you cut it with planes parallel to the y−axis.

(5) Compute $\displaystyle\int\int\int_W yz\, dV(x,y,z)$.

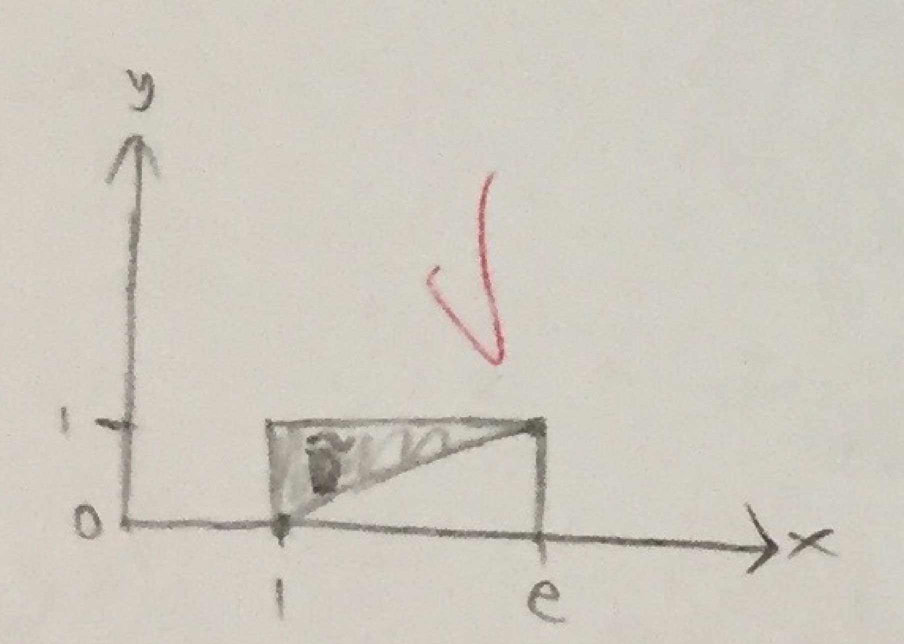
Hint: use the previously determined cross sections and use eventually polar coordinates in the yz−plane.

+0.5 1)



$$\text{Area}(D) = \int_1^e \ln x\, dx$$
$$= \cancel{(x\ln x - e^x)\Big|_1^e}$$
$$\cancel{e - e^e = e^{1-e}}$$

+0.5 2)

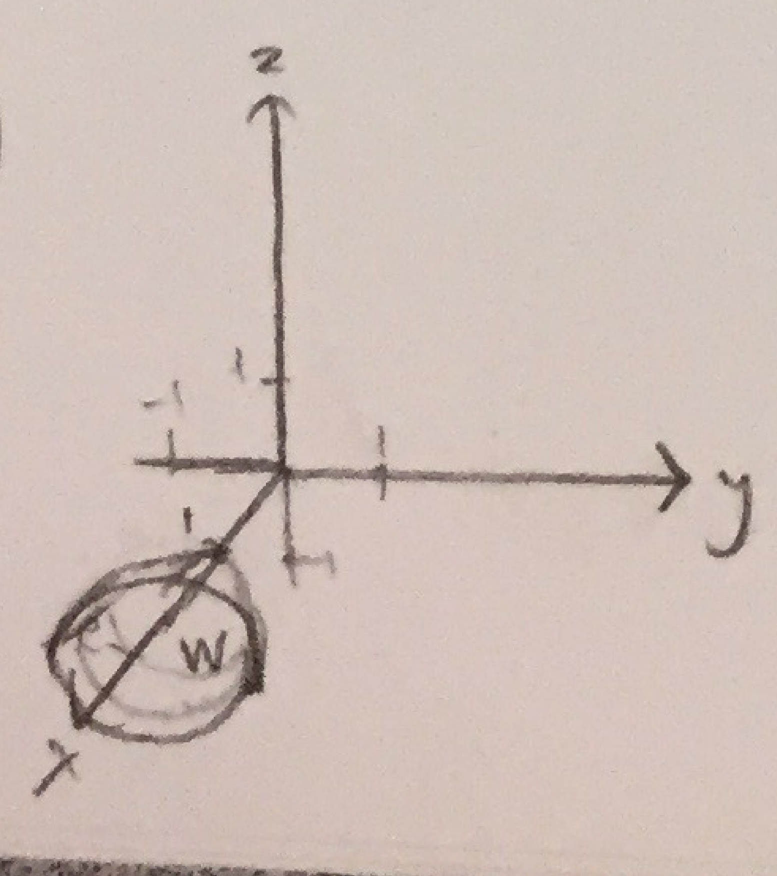


$$\text{Area}(\widetilde{D}) = (1\cdot e) - \int_1^e \ln x\, dx$$
$$= \cancel{e - e^{1-e}}$$
$$= \cancel{e^e}$$

+0 3)

$$\iint_D \frac{e^{2y}}{x}\, dA = \cancel{\int_1^e \int_0^1 \frac{e^{2y}}{x}\, dy\, dx} = \int_1^e \left(\frac{e^{2y}}{2x}\right)\Bigg|_{y=0}^{1} dx$$
$$= \int_1^e \frac{e^2-1}{2x}\, dx$$
$$= \frac{e^2-1}{2}(\ln x)\Big|_{x=1}^{e}$$
$$= \frac{e^2-1}{2} - 0 = \boxed{\frac{e^2-1}{2}}$$

+1 4)



Cross sections w.r.t. x-axis are disks of radius ln x.
Thus, the disk areas are πy^2. volume is the sum
(integral) of the disk areas from $x=1$ to $x=e$.
$$V(W) = \pi \int_1^e \int_0^{\ln x} \cancel{y^2}\, dy\, dx$$

Exercise 2 (6 points).

(1) Sketch the domains of integration and compute the following difference

$$\underbrace{\int_0^2\int_{x/2}^1 \cos(y^2)\,dy\,dx}_{D_1} - \underbrace{\int_0^1\int_x^1 \cos(y^2)\,dy\,dx}_{D_2}.$$

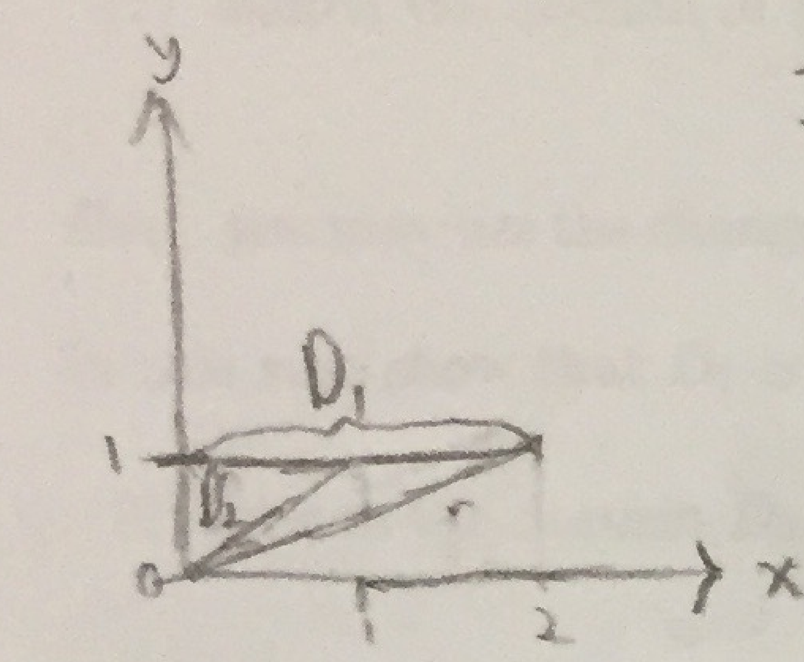
Hint: interchange the order of integration in both integrals. You may write the difference after as a single integral.

(2) Sketch the domain of integration, find an appropriate change of variable and compute

$$\iint_D \frac{1}{\cos^2(x^2+y^2)}\,dA(x,y),$$

where $D = \{\pi/6 \le x^2+y^2 \le \pi/4\}$.

1)



Integral 1 switch: y varies from 0 to 1, x varies from 0 to 2y

Integral 2 switch: y varies from 0 to 1, x varies from 0 to y

$$\int_0^1\int_0^{2y} \cos(y^2)\,dy\,dx - \int_0^1\int_0^{y} \cos(y^2)\,dy\,dx$$

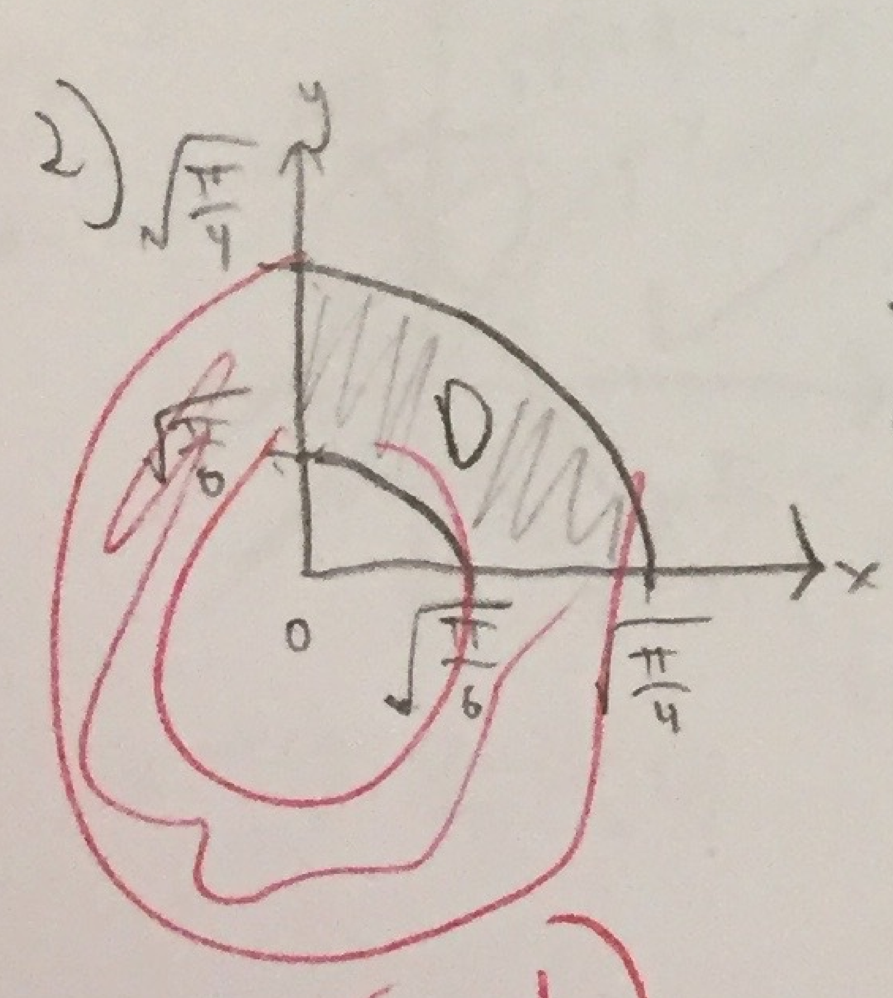
$$= \int_0^1 2y\cos(y^2)\,dy - \int_0^1 y\cos(y^2)\,dy$$

$$= 2\int_0^1 y\cos(y^2)\,dy - \int_0^1 y\cos(y^2)\,dy$$

$$= \int_0^1 y\cos(y^2)\,dy$$

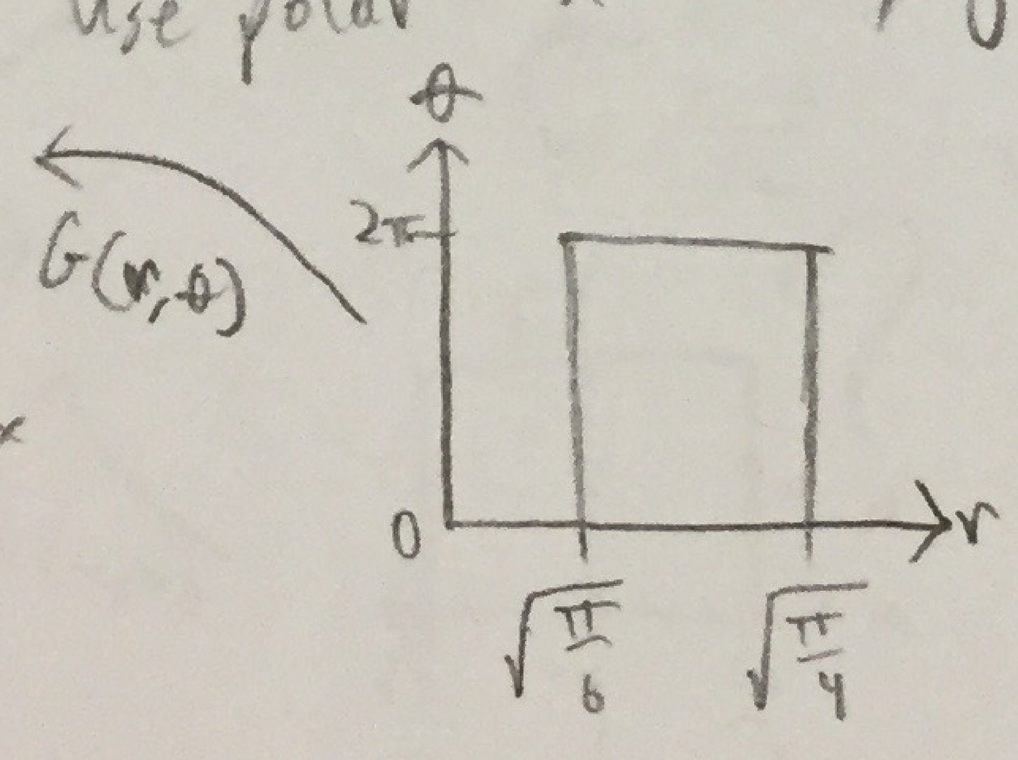
$$= \frac{1}{2}\sin(y^2)\Big|_0^1 = \boxed{\frac{1}{2}\sin(1)}$$

2)



(−1)

Use polar $x = r\cos\theta$, $y = r\sin\theta$, Jacobian $= r$, $x^2+y^2=r^2$



$$\iint_D \frac{1}{\cos^2(x^2+y^2)}\,dA_{(x,y)}$$

$$= \int_0^{2\pi}\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{4}}} \frac{r}{\cos^2(r^2)}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{4}}} r\sec^2(r^2)\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \tan(r^2)\Big|_{r=\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{4}}}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1-\frac{\sqrt{3}}{3}\right)d\theta$$

$$= \frac{1}{2}\left(1-\frac{\sqrt{3}}{3}\right)\theta\Big|_0^{2\pi} = \boxed{\pi - \frac{\pi\sqrt{3}}{3}}$$

Exercise 3 (7 points).

Let us consider the following disks $C_1 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ and $C_2 := \{(x,y) \in \mathbb{R}^2 : x^2+(y-\sqrt{2})^2 \leq 1\}$. We consider moreover the following domains:

- D_1 is bounded by the lines $y=x$, $y=\sqrt{2}-x$, $y=\sqrt{2}+x$ and $y=-x$;
- $D_2 = D_1 \cap C_1$, a circular sector of the disk C_1;
- $D_3 = D_1 \cap C_2$, a circular sector of the disk C_2;
- $D_4 = D_1 \setminus D_2$, the part of D_1 above D_2.
- $D_5 = C_1 \cap C_2$, the intersection of the disks C_1 and C_2;

Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = x^2+y^2$.

(1) Sketch the domain of D_1 and the disks C_1 and C_2, then compute $\iint_{D_1} f(x,y)\,\mathrm{d}A(x,y)$.

Hint: you may use the change of variables formula with the mapping $G(u,v) = \left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v, -\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v\right)$.

In this case show that D_1 is the image of the square $\widetilde{D}_1 = [-1,0] \times [0,1]$ through G.

(2) Sketch the domain D_2. Using eventually polar coordinates, compute $\iint_{D_2} f(x,y)\,\mathrm{d}A(x,y)$.

(3) Sketch the domain D_3. Using eventually polar coordinates, compute $\iint_{D_3} f(x,y)\,\mathrm{d}A(x,y)$.

(4) Sketch D_4 and compute $\iint_{D_4} f(x,y)\,\mathrm{d}A(x,y)$.

(5) Sketch D_5 and compute $\iint_{D_5} f(x,y)\,\mathrm{d}A(x,y)$.

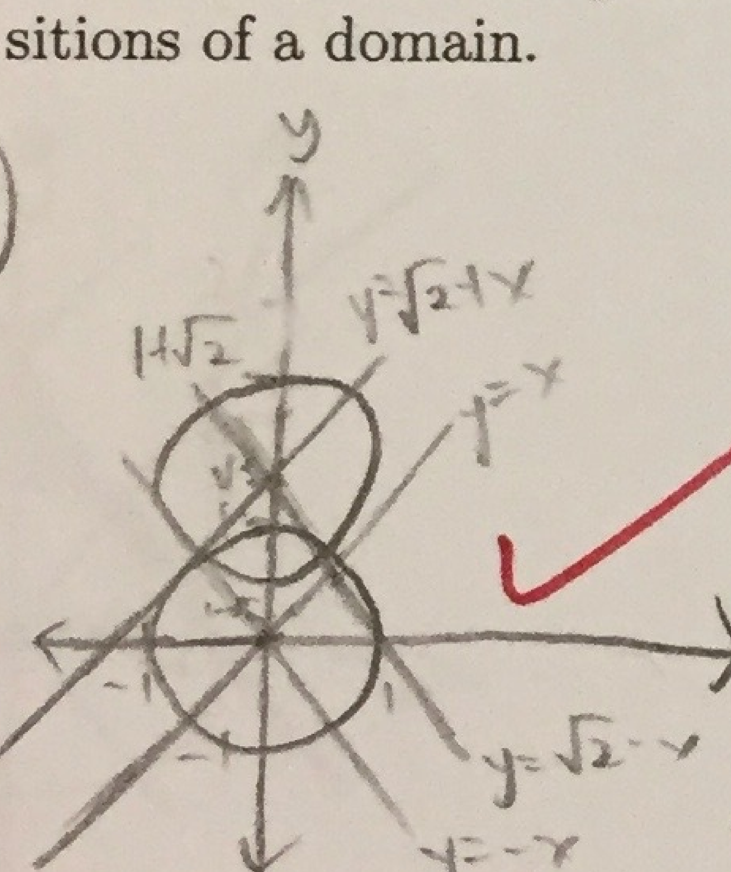
Hint: in the last two points you may use the additive property of the integral with respect to decompositions of a domain.

1p

1)

$G(u,v) = \left(\underbrace{\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v}_{x}, \underbrace{-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v}_{y}\right)$

$$\begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix}$$



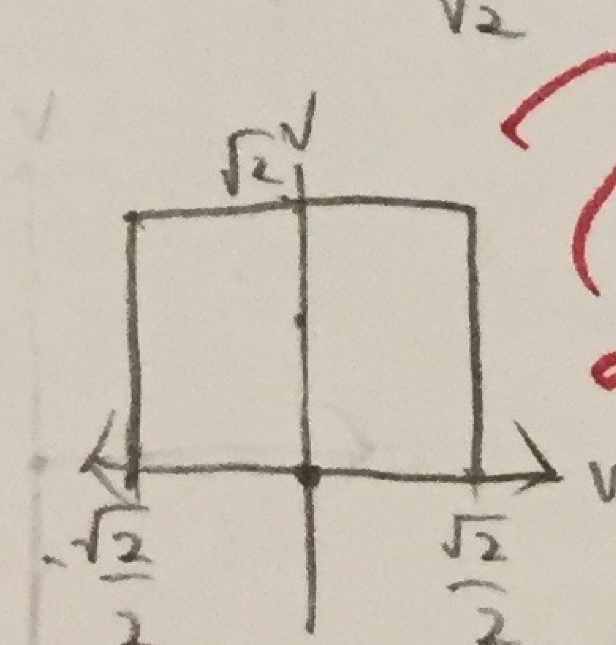
$x - y = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \frac{\sqrt{2}}{2}v$

$u = \frac{x-y}{\sqrt{2}}$ $\quad v = \frac{x+y}{\sqrt{2}}$

$Jac(G) = \frac{1}{2} + \frac{1}{2} = 1$

$G(u,v)$

?



$x \to (0, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, 0)$

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_0^{\sqrt{2}} 2u^2 + 2v^2 \, dv\, du$$

$$= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 2\sqrt{2} + 2u^2 \, du$$

not true!

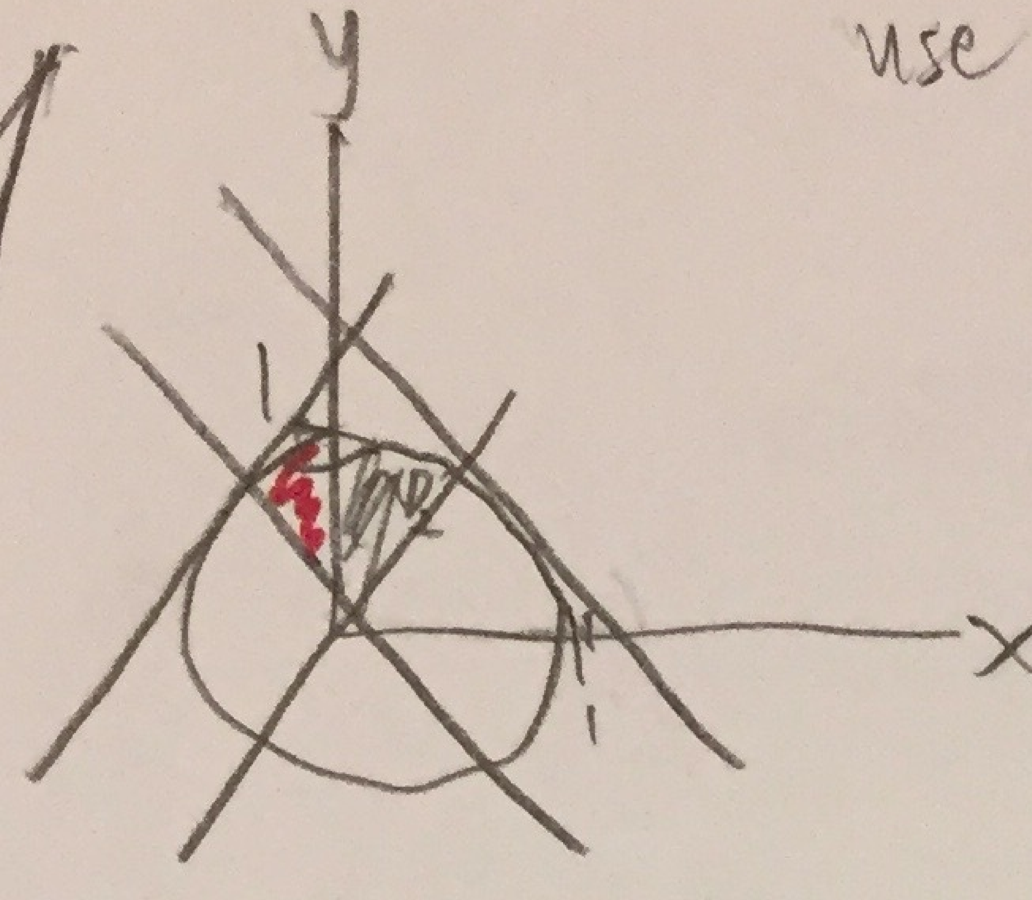
$$= 2\sqrt{2} + \frac{1}{2}\sqrt{2}$$

$$= \boxed{\frac{3\sqrt{2}}{2}}$$

cont

2) 8

0.5p



use polar

$\frac{3\pi}{4}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r^3\,dr\,d\theta$$

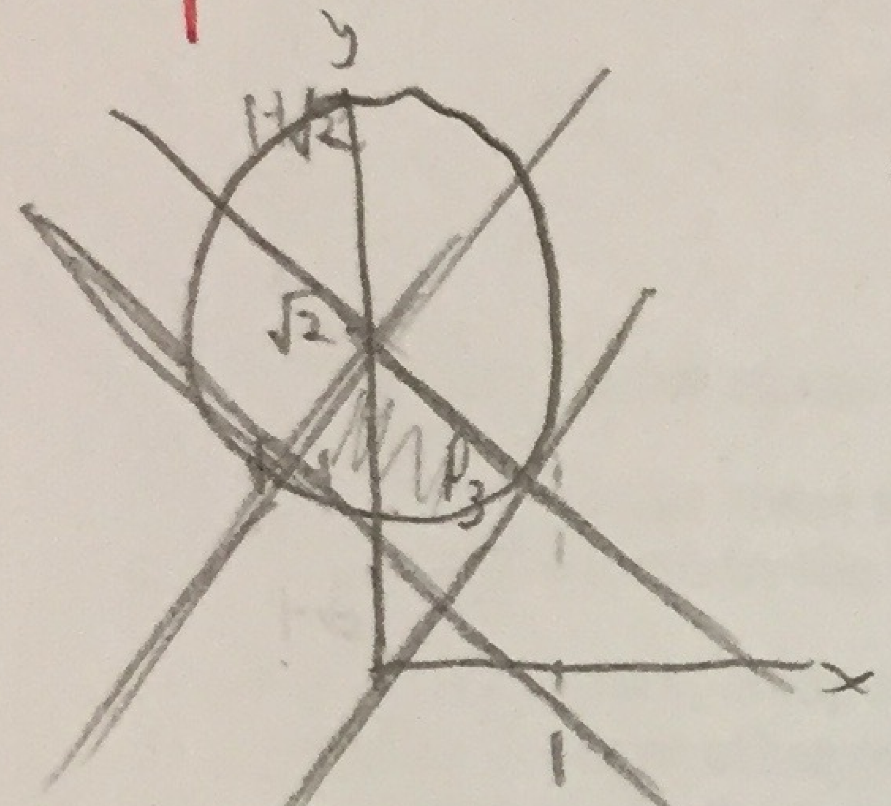
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^1 d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\,d\theta$$

$$= \frac{\pi}{8} - \frac{\pi}{16} = \boxed{\frac{\pi}{16}}$$

0.5p

3)



use polar

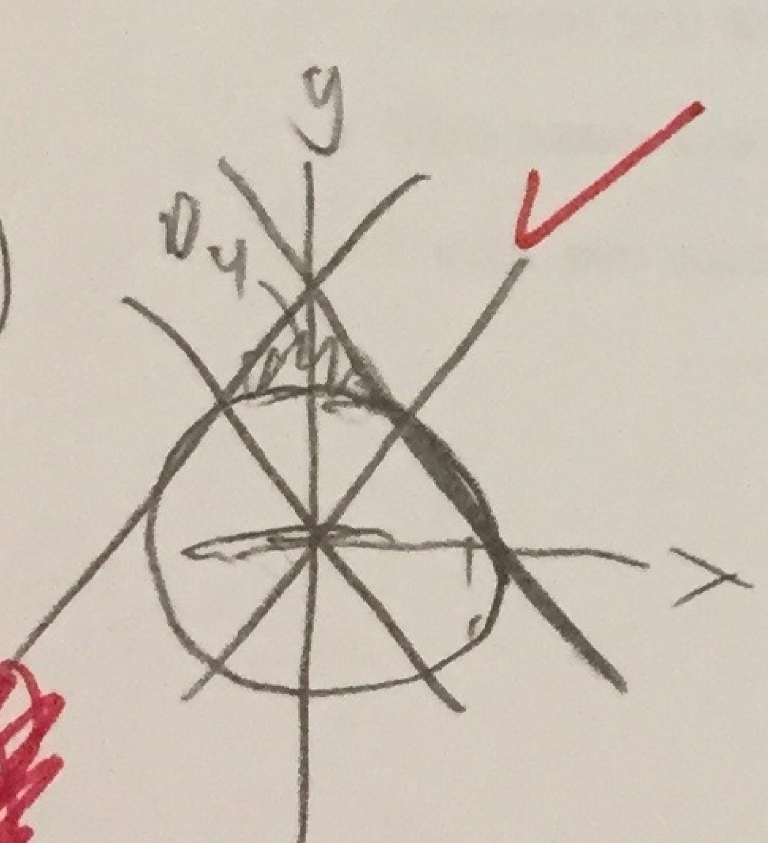
$$\int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} \int_0^1 r^3\,dr\,d\theta$$

$$= \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} \frac{1}{4}\,d\theta$$

$$= \frac{7\pi}{16} - \frac{5\pi}{16} = \boxed{\frac{\pi}{8}}$$

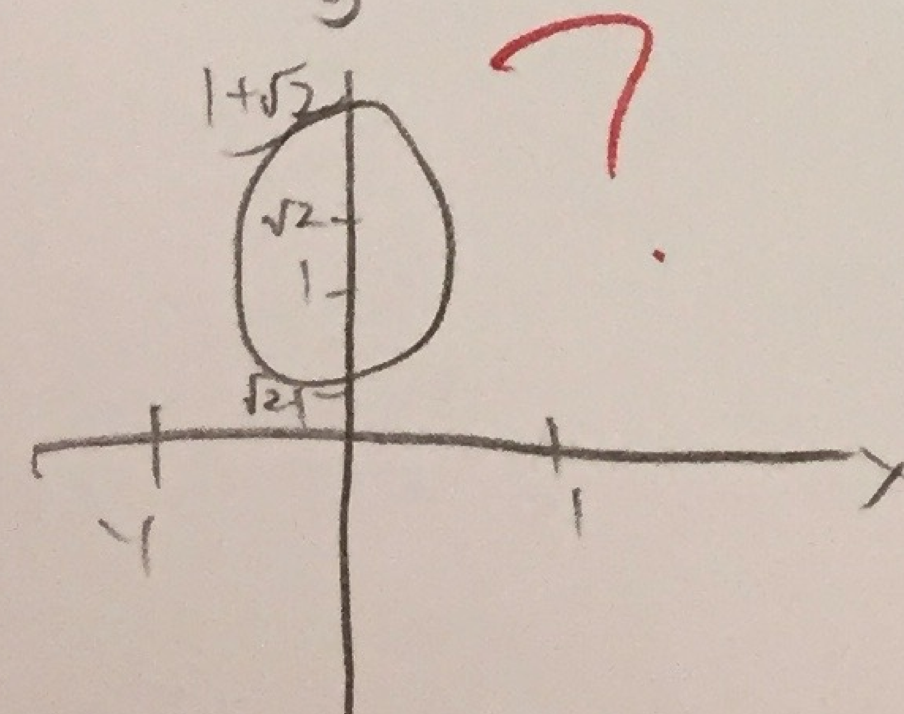
not really! You need other coordinates

4)



0p

5)



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