1	4
2	4
3	4
4	2.5
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MATH 32B Midterm II, Winter 2017

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

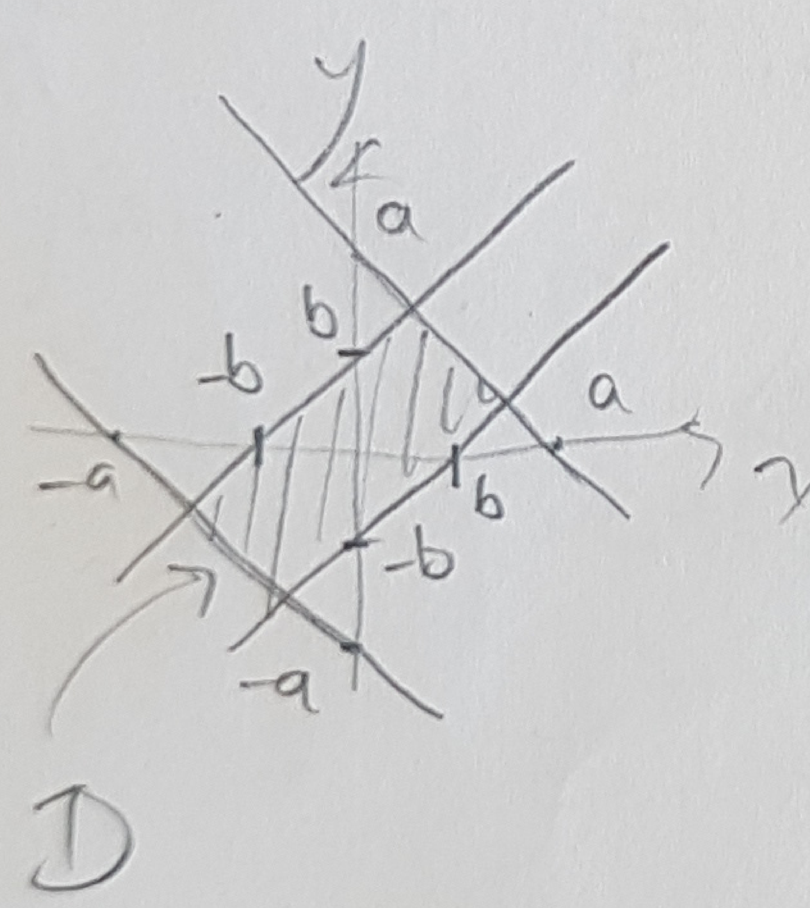
Evaluate the following integral

$$\iint_D x^2 + 2y^2\,dx\,dy.$$

Here D is the finite region in $\mathbb{R}^2$ bounded by the lines $x + y = a, x + y = -a, x - y = b, x - y = -b$, where a and b are two positive constants.

let $u = x+y \longrightarrow 2y = u - v \rightarrow y = \frac{1}{2}u - \frac{1}{2}v$

$v = (x-y) \rightarrow 2x = u + v \rightarrow x = \frac{1}{2}u + \frac{1}{2}v$



$$\text{Jacobian} = \left| \left[\det \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \right]^{-1} \right|$$

$$= \left| \left(\det \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right)^{-1} \right|$$

$$= \left| \frac{1}{-2} \right| = \frac{1}{2}$$

+2

$f(x,y) = x^2 + 2y^2$

$f(u,v) = (\frac{1}{2}u + \frac{1}{2}v)^2 + 2(\frac{1}{2}u - \frac{1}{2}v)^2$

$= \frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 + 2(\frac{1}{4}u^2 - \frac{1}{2}uv + \frac{1}{4}v^2)$

$= \frac{3}{4}u^2 - \frac{1}{2}uv + \frac{3}{4}v^2$

$\therefore$ new bounds

$x+y = a \rightarrow u = a$

$x+y = -a \rightarrow u = -a$

$x-y = b \rightarrow v = b$

$x-y = -b \rightarrow v = -b$

Flip

$$\iint_D x^2 + 2y^2 \, dx\,dy$$

$$= \int_{v=-b}^{b} \int_{u=-a}^{a} \left(\frac{3}{4}u^2 - \frac{1}{2}uv + \frac{3}{4}v^2 \right)\left(\frac{1}{2}\right) du\,dv$$

$$= \frac{1}{2} \int_{v=-b}^{b} \frac{1}{4}u^3 - \frac{1}{4}u^2 v + \frac{3}{4}v^2 u \Big|_{-a}^{a} \, dv$$

$$= \frac{1}{2} \int_{v=-b}^{b} \frac{1}{4}a^3 \cancel{- \frac{1}{4}a^2 v} + \frac{3}{4}v^2 a + \frac{1}{4}a^3 \cancel{+ \frac{1}{4}a^2 v} + \frac{3}{4}v^2 a \, dv$$

$$= \frac{1}{2} \int_{v=-b}^{b} \frac{1}{2}a^3 + \frac{3}{2}v^2 a \, dv$$

$$= \frac{1}{2} \left(\frac{1}{2}a^3 v + \frac{1}{2}a v^3 \Big|_{-b}^{b} \right)$$

$$= \frac{1}{2} \left(\frac{1}{2}a^3 b + \frac{1}{2}ab^3 + \frac{1}{2}a^3 b + \frac{1}{2}ab^3 \right)$$

$$= \boxed{\frac{1}{2}a^3 b + \frac{1}{2}ab^3}$$

+2

4

Problem 2. (4)

Find the line integral

$$\int_{C_1+C_2} yzdx + xzdy + xydz,$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,0,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

By observation, $\vec{F} = \langle yz, xz, xy \rangle$

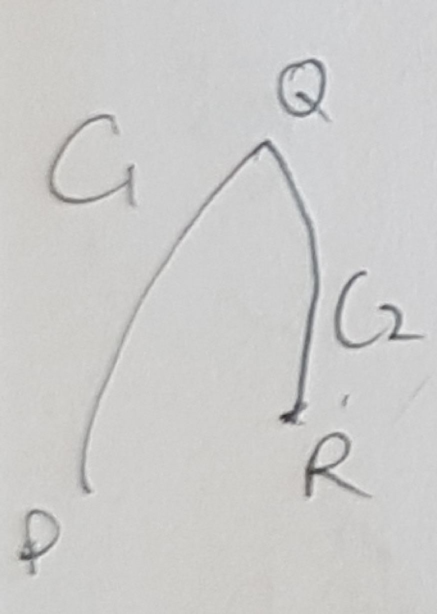
Also by observation, $\nabla f = \nabla(xyz)$

$= \langle yz, xz, xy \rangle$

$= \vec{F}$

$\therefore \vec{F} = \nabla f,\ f = xyz$

$\therefore \vec{F}$ is a conservative vector.



$\therefore \int_{C_1+C_2} \langle yz, xz, xy \rangle\, dxdydz$

$= xyz \Big|_P^R$ → because conservative vectors are path independent

$= [(0)(1)(1)] - [(1)(0)(1)]$

$= \boxed{0}$

4

Problem 3. (4)

Let $\mathbf{F} = (x^3 + y)\mathbf{i} + (x + z - \sin y)\mathbf{j} + (z^2 + y + \cos z)\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t^2$, $y = t^4$, $z = t^6$, $0 \le t \le 1$ with the orientation given by the parametrization.

i)

$\vec{F} = \langle x^3+y,\ x+z-\sin y,\ z^2+y+\cos z \rangle$

$\frac{\partial F_1}{\partial y} = 1 \qquad \frac{\partial F_1}{\partial z} = 0 \qquad \frac{\partial F_2}{\partial z} = 1$

$\frac{\partial F_2}{\partial x} = 1 \qquad \frac{\partial F_3}{\partial x} = 0 \qquad \frac{\partial F_3}{\partial y} = 1$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

Since x, y, z are (continuous) defined everywhere,

$\therefore$ $\boxed{\vec{F} \text{ is conservative}}$

ii) $V = \int F_1\,dx = \frac{1}{4}x^4 + yx + g(y,z)$

$V = \int F_2\,dy = xy + zy + \cos y + h(x,z)$

$V = \int F_3\,dz = \frac{1}{3}z^3 + zy + j(x,y) + \sin z$

By inspection,

$$\boxed{V = \frac{1}{4}x^4 + xy + zy + \frac{1}{3}z^3 + \cos y + \sin z + C}$$

constant

iii) F is conservative,

$\therefore \int_C F\,dr$ is path independent

$x = t^2,\ y = t^4,\ z = t^6,\ 0 \le t \le 1$

$\therefore \int_C F\,dr = \frac{1}{4}(t^2)^4 + (t^2)(t^4) + (t^6)(t^4) + \frac{1}{3}(t^6)^3 + \cos(t^4) + \sin(t^6) \Big|_0^1$

$\frac{3}{12} + \frac{4}{12}$

$\frac{7}{12} + \frac{12}{12} = \frac{19}{12}$

3

$= \frac{1}{4} + 1 + 1 + \frac{1}{3} + \cos 1 + \sin 1 - 1$

$$\boxed{= \frac{19}{12} + \cos(1) + \sin(1)}$$

Problem 4. (4)

Evaluate the surface integral $\int\int_S (3x^2 + 4y^2 + 5z^2)dS$ where S is the sphere given by the equation $x^2 + y^2 + z^2 = a^2$ where a is a positive constant.

Parametrization of sphere: $r(\theta, \phi) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle$

$\| r'(\theta,\phi) \| = a^2 \sin\phi \qquad r'(\theta,\phi) = a^2\sin\phi\, e_r$

$f(x,y,z) = 3x^2 + 4y^2 + 5z^2$

$f(r(\theta,\phi)) = 3a^2\sin^2\phi\cos^2\theta + 4a^2\sin^2\phi\sin^2\theta + 5a^2\cos^2\phi$

$$\therefore \iint_S f(x,y,z)\,dS = \iint f(r(\theta,\phi))\,\|r'(\theta,\phi)\|\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi}\int_{\theta=0}^{\theta=2\pi} 3a^4\sin^3\phi\cos^2\theta + 4a^4\sin^3\phi\sin^2\theta + 5a^4\sin\phi\,\boxed{\cancel{\cos\phi}}\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} 3a^4\sin^3\phi + a^4\sin^3\phi\sin^2\theta + 5a^4\sin\phi\cos\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi} 6a^4\pi\sin^3\phi + a^4\sin^3\phi\,\pi - \left(\frac{a^4\sin^3\phi}{4}\sin 2\theta\Big|_0^{2\pi}\right) + 0\,d\phi$$

$$= \int_{\phi=0}^{\pi} 7a^4\pi\sin^3\phi - 0\,d\phi$$

$$= 7a^4\pi\left(\frac{1}{3}\cos^3\phi - \cos\phi\right)\Big|_0^{\pi}$$

$$= 7a^4\pi\left(\frac{2}{3}\right) + 7a^4\pi\left(\frac{2}{3}\right)$$

$$= \boxed{\frac{28}{3}a^4\pi}$$

$\int \sin^3\phi$

$= -\cos + \frac{1}{3}\cos^3$

$\sin\phi - \cos^2\phi\sin\phi$

$= -\sin\phi(1+\cos^2\phi)$

$= \sin\phi(2\cos^2 + \sin^2)$

$\int \sin\phi$

$\sin\theta \Big|_0^{2\pi}$

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 - \sin^2$

$= 1 - 2\sin^2\theta$

$\cos^2 = \cos 2A$

$\frac{1-\cos 2A}{2}$

$1 - \frac{1}{3}$

$\frac{1}{3} - 1 = -\frac{2}{3}$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u, v) = (u, v, au^2 + bv^2)$ with domain $D = \{(u, v) | u^2 + v^2 \leq 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{N}$ pointing to the negative z-direction (that is the z-component of $\mathbf{N}$ is negative). Find

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} = < z, x^2, x^2 >$.

$G(u,v) = (u, v, au^2 + bv^2)$

$G_u = (1, 0, 2au)$

$G_v = (0, 1, 2bv)$

normal $\tilde{N} = G_u \times G_v$

$= (-2au, -2bv, 1)$

but have $z > 0$,

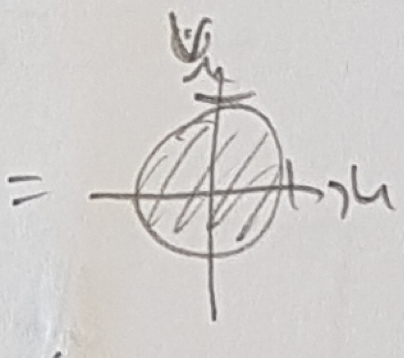
$\therefore \boxed{\vec{N} = -\tilde{N} = (2au, 2bv, -1)}$

$\iint_S \mathbf{F} \cdot d\mathbf{s}$

$= \iint_D F(u,v) \cdot \vec{N}(u,v)\, du\,dv$

$= \iint_D \langle au^2 + bv^2, u^2, u^2 \rangle \cdot \langle 2au, 2bv, -1 \rangle\, du\,dv$

$= \iint_D 2a^2u^3 + 2abuv^2 + 2bu^2v - u^2\, du\,dv$

But $D =$ 

$\therefore$ use polar coordinates

$\therefore$ $u = r\cos\theta$

$v = r\sin\theta$

$\therefore \iint_S F \cdot ds$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{r=1} \big(2a^2r^3\cos^3\theta + 2ab\cos\theta\sin^2\theta\, r^3 + 2br^3\cos^2\theta\sin\theta - r^2\cos^2\theta\big)\, r\, dr\, d\theta$

Flip.

$$= \int_0^{2\pi}\int_0^1 2a^2r^4\cos^3\theta + 2abr^4\cos\theta\sin^2\theta + 2br^4\cos^2\theta\sin\theta - r^3\cos^2\theta \; dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{5}a^2\cos^3\theta + \frac{2}{5}ab\cos\theta\sin^2\theta + \frac{2}{5}b\cos^2\theta\sin\theta - \frac{1}{4}\cos^2\theta \; d\theta$$

$$= \frac{2}{5}a^2\left(\sin\theta - \frac{1}{3}\sin^3\theta\right) + \frac{2}{5}ab\left(\frac{1}{3}\right)\sin^3\theta - \frac{2}{5}b\cos^3\theta\left(\frac{1}{3}\right) - \left(\frac{1}{4}\sin 2\theta - \frac{1}{2}\theta\right)\Bigg|_0^{2\pi}$$

$\frac{1}{4}$

$$= \frac{2}{5}a^2(0) + \frac{2}{5}ab\left(\frac{1}{3}\right)(0) - \frac{2}{5}b\left(\frac{1}{3}\right)(1-1) - 0 - \cancel{1}\pi$$

$1-1=0$

$\frac{1}{4}\pi$

$$= \boxed{+\frac{4}{15}b - \pi}$$

$\times -0.5$

~~sin2~~

$\sin 2A = 2\sin A$

$\cos 2A = \cos^2 - \sin^2$

$= 1 - 2\sin^2$

$2\cos^2 - 1$

$\frac{\cos 2A + 1}{2}$