Problem 5. (4)

Let S be a surface given by the parametric equation $r(u,v) = u\mathbf{i}+v\mathbf{j}+(u^2-2v^2)\mathbf{k}$. Find the equation of the tangent plane of S at the point $(1,1,-1)$.

$r_u = \langle 1, 0, 2u \rangle$

$r_v = (0, 1, -4v)$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & -4v \end{vmatrix}$$ [illegible]

$-2ui - vj + k$

At $(1,1)$ $\quad u=1 \quad v=1$

$-2i + 4j + k$

$-2(x-1)+4(y-1)+(z+1)=0$

1	4
2	4
3	4
4	4
5	4
T	20

MATH 32B Midterm II, Winter 2012

Name [redacted]

TA's Name and Section Number: Ben 3B

Problem 1. (4)

Evaluate the following line integral

$$\int_C 2y\,dx - x\,dy$$

by two different methods: (a) compute it directly; (b) use the Green theorem to compute it. Here C is the unit circle $x^2 + y^2 = 1$ with the counter clockwise orientation.

1a. $x = \cos\theta \quad dx = -\sin\theta$

$y = \sin\theta \quad dy = \cos\theta$

$\int_0^{2\pi} -2\sin^2\theta - \cos^2\theta$

$-\int_0^{2\pi} 2\sin^2\theta + \cos^2\theta$

$-\frac{1}{2}\int_0^{2\pi} 2 - 2\cos 2\theta + 1 + \cos 2\theta$

$-\frac{1}{2}\left[3\theta - \sin 2\theta\right]_0^{2\pi}$

$\boxed{-3\pi}$

1b. $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\iint_D (-1 - 2)\, dA$

$\int_0^{2\pi}\int_0^1 -3\, r\,dr\,d\theta$

$\pi\left[-3r^2\right]_0^1$

$\boxed{-3\pi}$

5. $\iint_S (x^2+y^2)\,ds \qquad S = x^2+y^2+z^2=a^2$

$$f(s) = (a\cos\theta\sin\phi)^2 + (a\sin\theta\sin\phi)^2$$
$$= a^2\sin^2\phi(\cos^2\theta + \sin^2\theta)$$
$$= a^2\sin^2\phi$$

$$\|n\| = a^2\sin\phi$$

$$\int_0^{2\pi}\int_0^{\pi} a^4\sin^3\phi\,d\theta$$

$$= 2\pi a^4\left[\frac{\cos^3\theta}{3} - \cos\theta\right]\Big|_0^{\pi}$$

$$= 2\pi a^4\left[\left(-\tfrac{1}{3}+1\right)-\left(\tfrac{1}{3}-1\right)\right]$$

$$= 2\pi a^4\left[4/3\right]$$

$$= \frac{8}{3}\pi a^4$$

$x^2+y^2+z^2=z \qquad z=\sqrt{x^2+y^2}$

6. $\iiint_E z\,dV$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{2\pi}{4}\int_0^{\pi/4}\cos^5\phi\sin\phi\,d\phi$$

$$= \frac{1}{6}\cdot\frac{\pi}{2}\left[-\cos^6\phi\right]_0^{\pi/4}$$

$$= \frac{\pi}{12}\left[1-\left(\frac{1}{\sqrt{2}}\right)^6\right]$$

$$= \frac{\pi}{12}\left[1-\frac{1}{8}\right] = \boxed{\frac{7\pi}{96}}$$

Problem 3. (4)

Find the line integral

$$\int_C x^3 dx + y^2 dy + z dz,$$

where C is the line segment connecting the point $P = (0, 0, 1)$ and $Q = (1, 2, 4)$, and C is oriented by the direction from P to Q.

$r(t) = (1-t)r_0 + t r_1$

$r(t) = (1-t)\langle 0,0,1\rangle + t\langle 1,2,4\rangle$

$x = t \qquad \frac{dx}{dt} = 1$

$y = 2t \qquad \frac{dy}{dt} = 2$

$z = 1 - t + 4t \Rightarrow 1 + 3t \qquad \frac{dz}{dt} \cdot 3$

$\int_0^1 t^3 + 8t^2 + 3 + 9t \, dt$

$\left[\frac{t^4}{4} + \frac{8}{3}t^3 + 3t + \frac{9}{2}t^2\right]_0^1$

$\frac{1}{4} + \frac{8}{3} + 3 + \frac{9}{2}$

$\frac{3}{12} + \frac{32}{12} + \frac{36}{12} + \frac{54}{12}$

$= \boxed{\frac{125}{12}}$

3. $F = \langle 2x+3y, 3x-10y, 4z \rangle$

$$\text{Curl}(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 2x+3y & 3x-10y & 4z \end{vmatrix} = \langle 0-0, 0-0, 3-3 \rangle = 0$$

Conservative!

$\partial x f = 2x+3y$

$f = x^2 + 3xy + c(y,z)$

$\partial y f = 3x - 10y = 3x + c'(y,z)$

$c(y,z) = -5y^2 + c(z)$

$f = x^2 + 3xy - 5y^2 + 2z^2 + k$

$\partial z = 4z = c'(z)$

$c(z) = 2z^2 + k$

$c(t) = \langle t, t^2, t^3 \rangle$

$f(c(1)) - f(c(0))$

$= 1 - 0 = 1$

4 $r(u,v) = (u, v, u^2 - 2v^2)$

$T_u = \langle 1, 0, 2u)$

$T_v = \langle 0, 1, -4v)$

$N(u,v) = \langle -2u \; -4v, 1 \rangle$

$u = v = 1$

$N(1,1) = \langle -2, 4, 1 \rangle$

$\langle -2, -2, 1 \rangle \cdot \langle x-1, y-1, z+1 \rangle = 0$

$-2(x-1) + 4(y-1) + (z+1) = 0$

$-2x + 4y + z = +1$

Problem 4. (4)

Let $F = (2x + 3y)\mathbf{i} + (3x - 10y)\mathbf{j} + 4z\mathbf{k}$ defined on $\mathbb{R}^3$.
(i) Decide whether or not F is conservative.
(ii) If F is conservative, find the potential function f, such that $F = \nabla f$.
(iii) Compute the line integral $\int_C F \cdot dr$, where C is given by the parametric equation $x = t$, $y = t^2$, $z = t^3$ $0 \leq t \leq 1$ with the orientation given by the parametrization.

4i. $$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & 3x-10y & 4z \end{vmatrix} = (0-0)i - (0-0)j + (3-3)k$$

$$= 0i - 0j + 0k$$

since curl F is $= 0$, it is $\boxed{\text{conservative}}$

4ii. $$\frac{\partial F}{\partial x} = \int 2x+3y\,dx = x^2+3xy+C(y,z)$$

$$\frac{\partial F}{\partial y} = 3x + C'(y,z) = 3x - 10y$$

$$C'(y,z) = -10y$$

$$C(y,z) = -5y^2 + C(z)$$

$$\frac{\partial F}{\partial z} = C'(z) = 4zk$$

$$C(z) = 2z^2$$

$$\boxed{F = x^2 + 3xy - 5y^2 + 2z^2 + K}$$

4iii $$\int_0^1 t^2 + 3t^3 - 5t^4 + 2t^6$$

$$F(1) - F(0)$$

$$(1+3-5+2) - (0+0-0+0)$$

$$1 - 0$$

$$\boxed{1}$$

1 Evaluate: $\int_C 2y\,dx - x\,dy$ C: $x^2+y^2=1$ ccw

$r(\theta) = (\cos\theta, \sin\theta)$
$r'(\theta) = (-\sin\theta, \cos\theta)$

$$\int_0^{2\pi} -2\sin^2\theta - \cos^2\theta \, d\theta$$

$$= -\int_0^{2\pi} 2\sin^2\theta + \cos^2\theta \, d\theta$$

$$= -\frac{1}{2}\left[3\theta + \cos 2\theta\right]\Big|_0^{2\pi}$$

$$= -\frac{1}{2}(3)(2\pi) = \boxed{-3\pi}$$

2. $\int_C x^3dx + y^2dy + z\,dz$ P $(0,0,1) \to$ Q $(1,2,4)$

$r(t) = (1-t)P + (t)Q$
$= (t, 2t, 1+3t)$
$r'(t) = (1, 2, 3)$

$F(x,y,z) = (x^3, y^2, z)$
$F(r(t)) = (t^3, 4t^2, 1+3t)$

$F(r(t)) \cdot r'(t) = t^3 + 8t^2 + 9t + 3$

$$\int_0^1 t^3 + 8t^2 + 9t + 3\,dt = \left[\frac{t^4}{4} + \frac{8}{3}t^3 + \frac{9}{2}t^2 + 3t\right]\Big|_0^1$$

$$= \left[\frac{1}{4} + \frac{8}{3} + \frac{9}{2} + 3\right] = \left[\frac{3+32+54+36}{12}\right] = \frac{125}{12}$$

Problem 2. (4)

Find

$$\iiint_E z \, dx dy dz,$$

where E is bounded by the sphere $x^2 + y^2 + z^2 = z$ and the cone $z = \sqrt{x^2 + y^2}$.

$\rho \cos\varphi = \rho^2$

$\rho = \cos\varphi$

$\rho \cos\varphi = \sqrt{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}$

$\rho \cos\varphi = \rho \sin\varphi$

$\cos\varphi = \sin\varphi$

$\varphi = \frac{\pi}{4}$ ✓

$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\varphi} \rho \cos\varphi \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$2\pi \int_0^{\frac{\pi}{4}} \int_0^{\cos\varphi} \rho^3 \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\frac{\pi}{2} \int_0^{\frac{\pi}{4}} \left[\rho^4\right]_0^{\cos\varphi} \cos\varphi \sin\varphi \, d\varphi$

$\frac{\pi}{2} \int_0^{\frac{\pi}{4}} \cos^5\varphi \sin\varphi \, d\varphi$

$\frac{\pi}{12} \left[-\cos^6\varphi\right]_0^{\frac{\pi}{4}}$

$\frac{\pi}{12} \left[-\left(\frac{\sqrt{2}}{2}\right)^6 + 1\right]$

$\frac{\pi}{12} \left[-\frac{8}{64} + \frac{64}{64}\right]$

$\frac{\pi}{12} \left(\frac{56}{64}\right) = \frac{7\pi}{96}$

$\boxed{\frac{7\pi}{96}}$ ✓