1	3
2	2
3	3
4	1.5
5	3
T	12.5

MATH 32B Midterm I, Winter 2017

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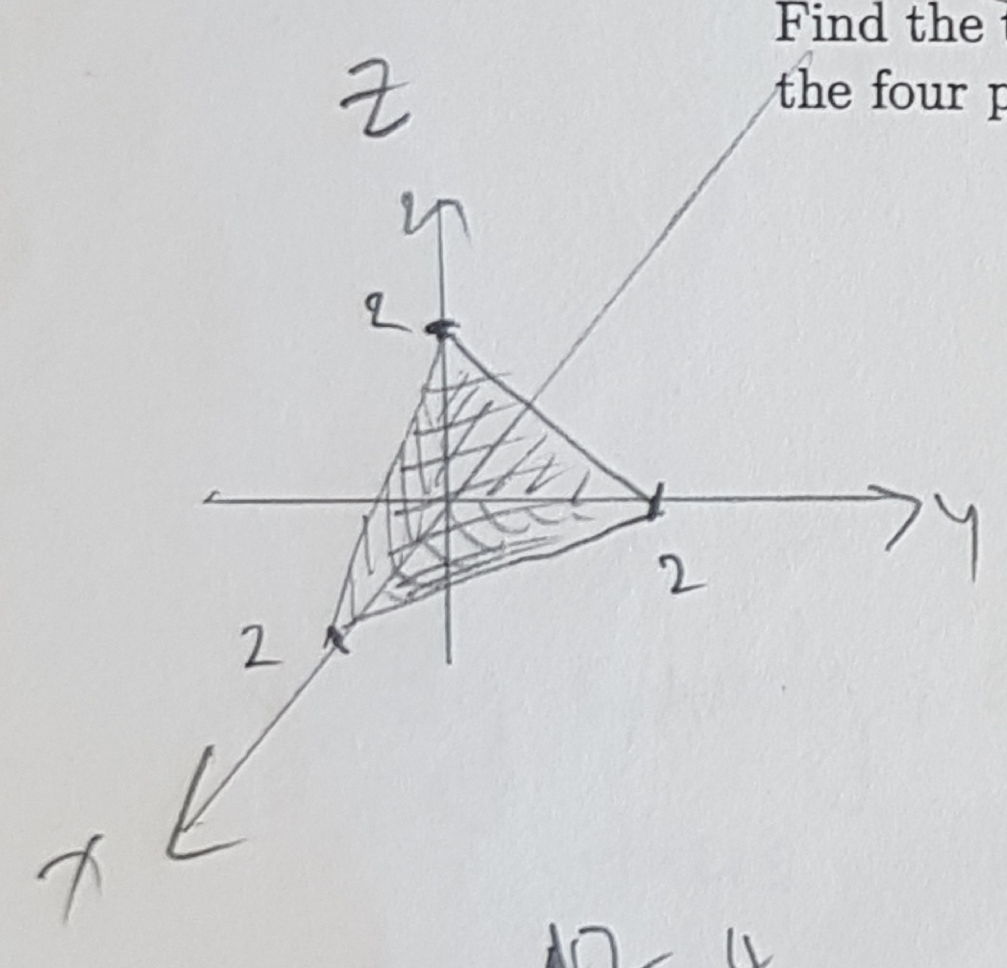
Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the triple integral $\int\int\int_E x\,dx\,dy\,dz$. Here E is the finite region bounded by the four planes $x=0,\ y=x,\ z=0$ and $x+y+z=2$.



$x+y+z=2$

$x=2-y-z$

$$\iiint_E x \; dx\,dy\,dz$$

$$= \int_{z=0}^{2} \cancel{\int_{y=0}^{2-z}} \int_{x=0}^{2-y-z} x \; dx\,dy\,dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{2-z} \frac{1}{2}x^2 \Big|_{x=0}^{2-y-z} dy\,dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{2-z} \frac{1}{2}(4-4y-4z+2yz+y^2+z^2)\,dy\,dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{2-z} (2-2y-2z+yz+\tfrac{1}{2}y^2+\tfrac{1}{2}z^2)\,dy\,dz$$

$$= \int_{z=0}^{2} 2y - y^2 - 2zy + \tfrac{1}{2}zy^2 + \tfrac{1}{6}y^3 + \tfrac{1}{2}z^2y \Big|_{y=0}^{2-z} dz$$

$$= \int_{z=0}^{2} 4-2z-(4-4z+z^2) - 4z + 4z^2 + \tfrac{1}{2}z(4-4z+z^2) + \tfrac{1}{6}(8-12z+6z^2-z^3) + \tfrac{1}{2}(2z-z^2)\; dz$$

$$= \int_{z=0}^{2} \tfrac{4}{3} - z + \tfrac{3}{2}z^2 + \tfrac{1}{3}z^3 \; dz$$

$$= \tfrac{4}{3}z - \tfrac{1}{2}z^2 + \tfrac{1}{2}z^3 + \tfrac{1}{12}z^4 \Big|_0^2$$

$$= \tfrac{8}{3} - 2 + \tfrac{8}{2} + \tfrac{4}{3} = \boxed{6}$$

$2+\frac{4\cdot 2}{3}=4$

~~Rough~~ working.

$\frac{12}{3}-\cancel{4}$

$(2-z)^2$ $\quad 2-\frac{8}{2}$

$= (4-4z+z^2)(2-z)-2$

$= 8-4z-8z+4z^2+2z^2-z^3$

$2z^2-\frac{1}{2}z^2$

$= \frac{4}{2}-\frac{1}{2}$

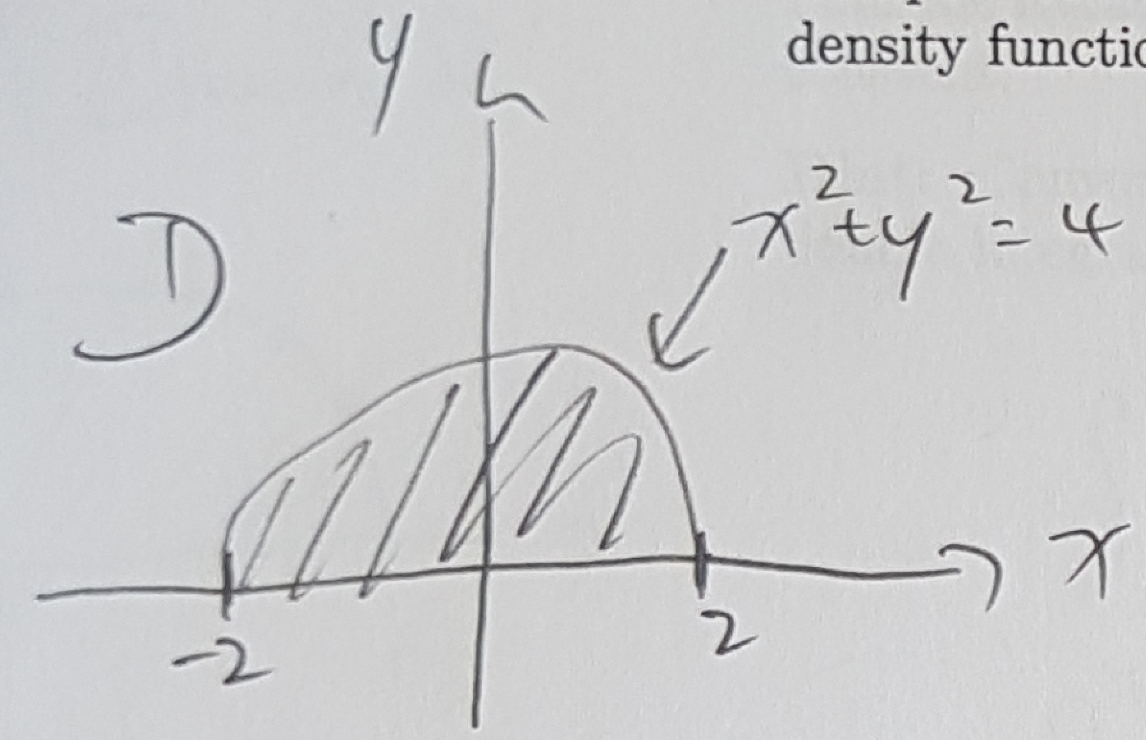
$\frac{1}{2}z^3 - \frac{1}{6}z^3$

$\frac{3}{6}\ \frac{1}{8}z^3$

Problem 2. (4)

Compute the center of of mass of $D = \{(x, y)|\ x^2 + y^2 \le 4,\ y \ge 0\}$. Here the density function $\rho(x, y) = x^2 + y^2$.

D



$$\text{Total mass} = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} x^2+y^2 \, dy\, dx$$

$$= \int_{r=0}^{r=2} \int_{\theta=0}^{\pi} r\,(r)\, d\theta\, dr$$

dx

$$= \int_{r=0}^{2} r^2\theta \Big|_0^{\pi} dr$$

$$= \int_{r=0}^{2} \pi r^2 \, dr$$

$$= \tfrac{1}{3}\pi r^3 \Big|_0^2$$

$$= \tfrac{8}{3}\pi$$

By symmetry, since $x=-x$ for the domain and $\rho(x,y)$,
thus $\bar{x}_{cm} = 0$

+1

$$\bar{y}_{cm} = \frac{1}{\text{Total mass}} \cdot \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} y(x^2+y^2)\, dy\, dx$$

S|θ
T|C

$$= \frac{3}{8\pi} \cdot \int_{r=0}^{2} \int_{\theta=0}^{\pi} r\sin\theta\,(r)\,(r)\, d\theta\, dr$$

$$= \frac{3}{8\pi} \cdot \int_{r=0}^{2} -r^3\cos\theta \Big|_0^{\pi} dr$$

+1

$$= \frac{3}{8\pi} \int_{r=0}^{2} 2r^3\, dr$$

$$= \frac{3}{8\pi} \cdot \left(\frac{1}{2} r^4 \Big|_0^2\right)$$

$$= \frac{3}{8\pi}(8) = \frac{3}{\pi}$$

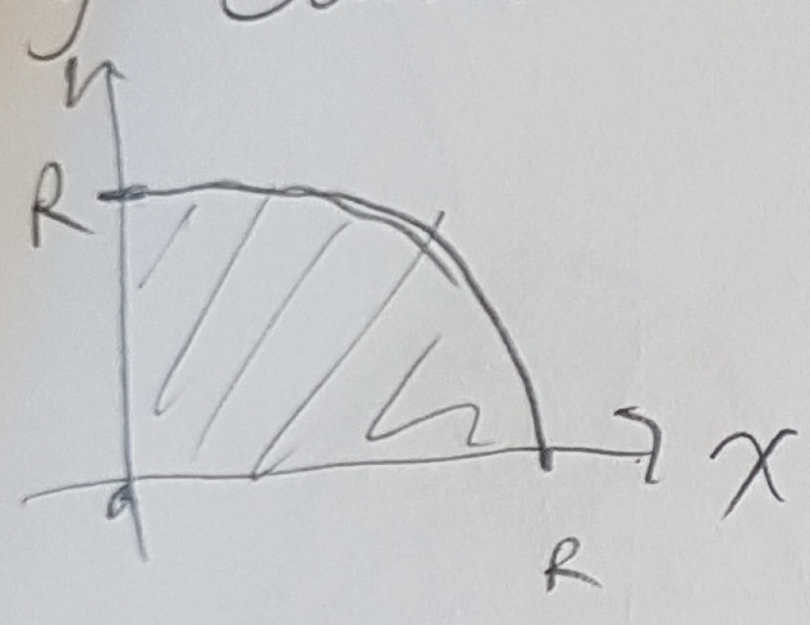
$\therefore$ center of mass $= \boxed{\left(0, \frac{3}{\pi}\right)}$

Problem 3. (4)

Find the iterated integral $\int_0^R \int_0^{\sqrt{R^2-x^2}} sin^2((x^2+y^2))dydx$. Here R is a positive constant.

Hint: Convert the iterated integral into a double integral and evaluate the double integral.

Domain D



$$\int_0^R \int_0^{\sqrt{R^2-x^2}} \sin^2((x^2+y^2))\,dydx$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{r=R} \sin^2(r^2)\; r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{r=R} r - r\cos(2r^2)\; dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{2}r^2 - \frac{1}{4}\sin(2r^2)\Big|_0^R \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{2}R^2 - \frac{1}{4}\sin(2R^2)\; d\theta$$

$$= \frac{1}{2}R^2\theta - \frac{1}{4}\sin(2R^2)\,\theta \Big|_{\theta=0}^{\pi/2}$$

$$\boxed{= \frac{\pi R^2}{4} - \frac{\pi}{8}\sin(2R^2)}$$

X 3

$\rho^2 - \rho\cos\phi$

$\rho^2 - z$

$x^2+y^2=1$

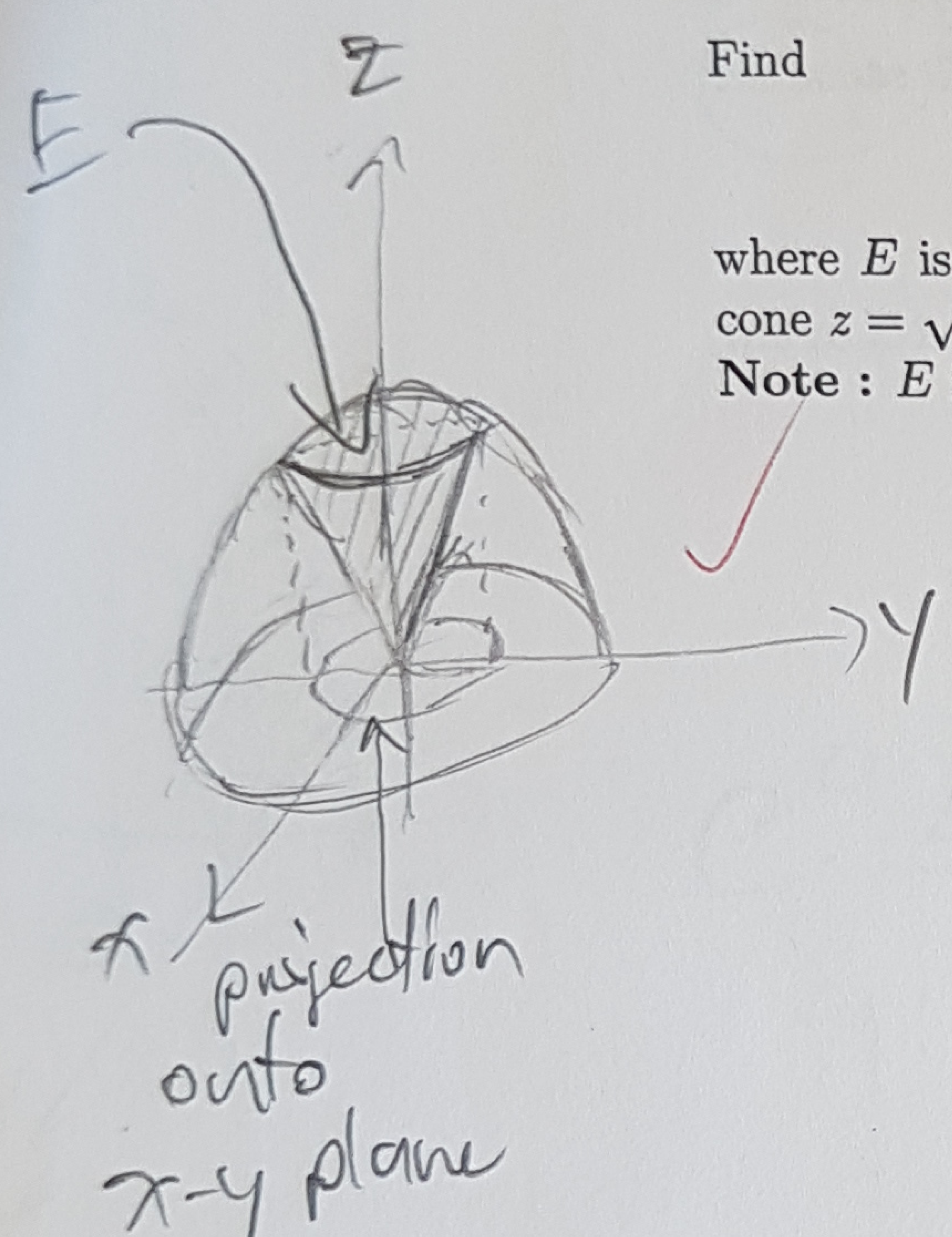
Problem 4. (4)

Find

$$\iiint_E x^2+y^2\,dxdydz,$$

where E is the finite solid bounded by the sphere $x^2+y^2+z^2 = 2z$ and the cone $z=\sqrt{x^2+y^2}$

Note : E is inside both the sphere and the cone.



equate θ with ρ

$x^2+y^2+z^2=2z$

$z=\sqrt{x^2+y^2}=\sqrt{2z-z^2}$

$z^2=2z-z^2$

$2z^2=2z$

$z=1$

$$\iiint_E x^2+y^2\,dxdydz$$

$$= \int_{\rho=0}^{\sqrt{2z}} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} (\rho^2-\rho\cos\phi)\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\sqrt{2\rho\cos\phi}} \cancel{\rho^4\sin\phi - \rho^3\cos\phi\sin\phi}\; d\rho\,d\phi\,d\theta.$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4}$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} 2$$

$z=\rho\cos\phi$

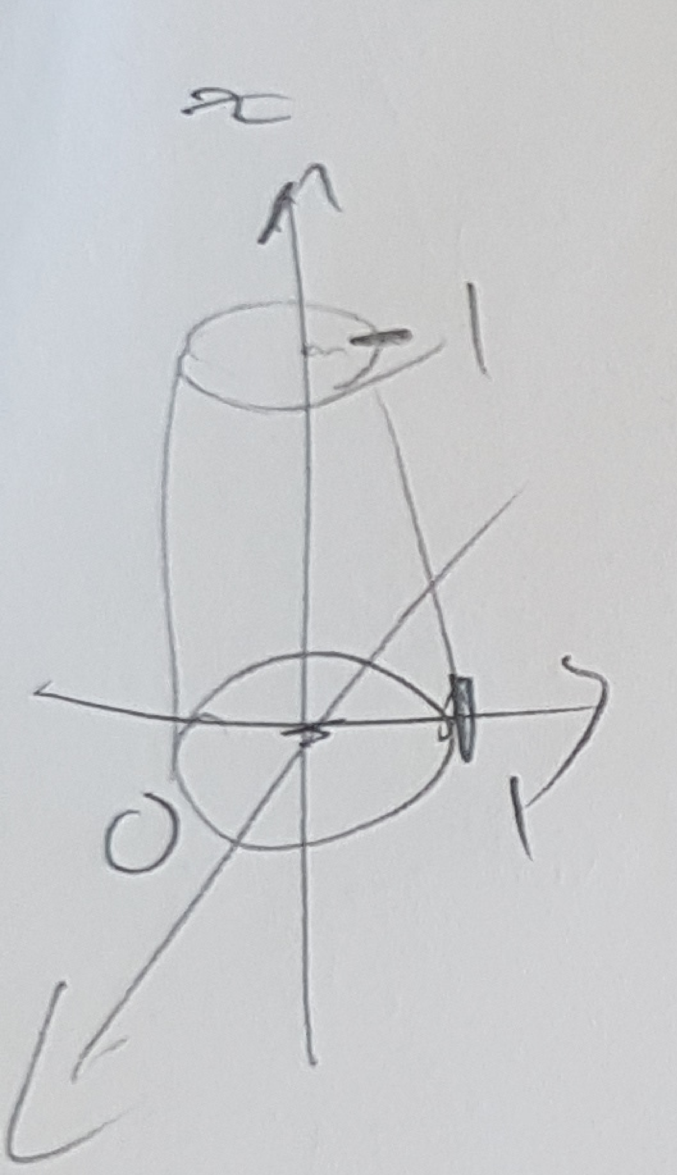
3

Problem 5. (4)

Estimate the following integral

$$\iiint_E e^{\cos(x^2+y^2)\cdot \sin z}\, dV,$$

where E is the region inside the cylinder $x^2+y^2=1$ with $0 \le z \le 1$.



$$\iiint_E e^{\cos(x^2+y^2)\cdot \sin z}\, dV$$

$$e^0 \le e^{\cos(x^2+y^2)\cdot \sin z} \le e^1$$

$$1 \le e^{\cos(x^2+y^2)\cdot \sin z} \le e^{\sin(1)}$$

$$\therefore \iiint_E 1\, dV \le \iiint_E e^{\cos(x^2+y^2)\cdot \sin z}\, dV \le \iiint_E e\, dV$$

$$\int_{z=0}^1 \int_{r=0}^1 \int_{\theta=0}^{2\pi} 1\, r\, dV \le \text{area} \le \int_{z=0}^1 \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} e r\, dV$$

$$\int_0^1 \int_0^1 \theta r \Big|_0^{2\pi} dr\, dz \le \text{area} \le \int_0^1 \int_0^1 2\pi e r\, dr\, dz$$

$$\int_0^1 \int_0^1 2\pi r\, dr\, dz \le \text{area} \le \int_0^1 e\pi r^2 \Big|_0^1 dz$$

$$\int_0^1 \pi r^2 \Big|_0^1 dz \le \text{area} \le \int_0^1 e\pi\, dz$$

$$\pi z \Big|_0^1 \le \text{area} \le e\pi z \Big|_0^1$$

$$\boxed{\pi \le \text{~~area~~} \le e\pi}$$