4. $\int_0^K \int_{-\sqrt{K^2-x^2}}^{\sqrt{K^2-x^2}} \sin(x^2+y^2)\,dy\,dx$ $\qquad 0 \le \theta \le 2\pi$

$0 \le r \le K$

$$\int_0^{\pi}\int_0^{K} \sin(r^2)\, r\, dr\, d\theta$$

$= \pi\left[-\frac{1}{2}\cos(r^2)\right]_0^K$

$= \frac{\pi}{2}\left[\cos(r^2)\right]_K^0$

$= \frac{\pi}{2}\left[1-\cos(K^2)\right]$ ✓

5a $\iiint e^{\sin x\cos y\sin z}\,dV$

$$-1 \le \sin x\cos y\sin z \le 1$$

$$e^{-1} \le e^{\sin x\cos y\sin z} \le e^{1}$$

$$4\pi R^2 \cdot e^{-1} \iiint e^{\sin x\cos y\sin z} \le e \cdot 4\pi R^2$$

$$V = \int_{-2}^{2}\int_0^{2\pi}\int_0^{R} r\,dr\,d\theta\,dz = (4)(2\pi)\cdot \frac{r^2}{2}\Big|_0^R = 8\pi\cdot\frac{R^2}{2} = \boxed{4\pi R^2}$$

5b $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2\sin\phi \cdot \rho\cos\phi\,d\rho\,\cancel{d\phi}\,d\theta$

$= 2\pi\int_0^{\pi/4} \frac{1}{4}\cdot\cos^5\phi\sin\phi\,d\phi$

$= \frac{\pi}{2}\left[-\frac{u^6}{6}\right]_0^{\pi/4}$

$= -\frac{\pi}{12}\left[\cos^6\phi\right]_0^{\pi/4}$

$= +\frac{\pi}{12}\left[1-\left(\frac{\sqrt{2}}{2}\right)^6\right]$

$= \frac{\pi}{12}\left[1-\frac{1}{8}\right] = \frac{\pi}{12}\left(\frac{7}{8}\right) = \boxed{\frac{7\pi}{96}}$

3. $\rho(x,y) = y$

$$\text{Mass} = \iint y \, dA$$

$$= \int_0^1 \int_{y-1}^{1-y} y \, dx\,dy = \int_0^1 yx \Big|_{y-1}^{1-y} = \int_0^1 y\big((1-y)-(y-1)\big)\,dy$$

$$= \int_0^1 y(2-2y)\,dy$$

$$= \int_0^1 2y - 2y^2 \, dy = y^2 - \frac{2}{3}y^3 \Big|_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

$$x_{cm} \; \frac{1}{M} \iint x \, \rho(x,y)\, dA$$

$$= 3\int_0^1 \int_{y-1}^{1-y} xy \, dx\,dy = 3\int_0^1 \frac{1}{2} y x^2 \Big|_{y-1}^{1-y} dy$$

$$= \frac{3}{2}\int_0^1 \underbrace{(1-y)^2 - (y-1)^2}\, dy \quad \to (y^2 - 2y + 1) - (y^2 - 2y + 1) = 0$$

$$y_{cm} = \frac{1}{M} \iint y^2 \, dA$$

$$= 3\int_0^1 \int_{y-1}^{1-y} y^2 \, dx\,dy = 3\int_0^1 y^2 x \Big|_{y-1}^{1-y} dy = 3\int_0^1 y^2\big((1-y) - (y-1)\big)\,dy$$

$$= 3\int_0^1 y^2(2-2y)\,dy$$

$$= 3\int_0^1 2y^2 - 2y^3 \, dy = 3\left[\frac{2}{3}y^3 - \frac{2}{4}y^4\right]_0^1 = 3\left(\frac{8}{12} - \frac{6}{12}\right] = 3\left[\frac{2}{12}\right] = 3 \cdot \frac{1}{6} = \boxed{\frac{1}{2}}$$

① $\displaystyle\int_0^1\int_0^{1-x}\int_0^{2-2x-2y} z\,dz\,dy\,dx$

$= \displaystyle\frac{1}{2}\int_0^1\int_0^{1-x} z^2\Big|_0^{2-2x-2y}$

$= \displaystyle\frac{1}{2}\int_0^1\int_0^{1-x} (2-2x-2y)^2\,dy\,dx$

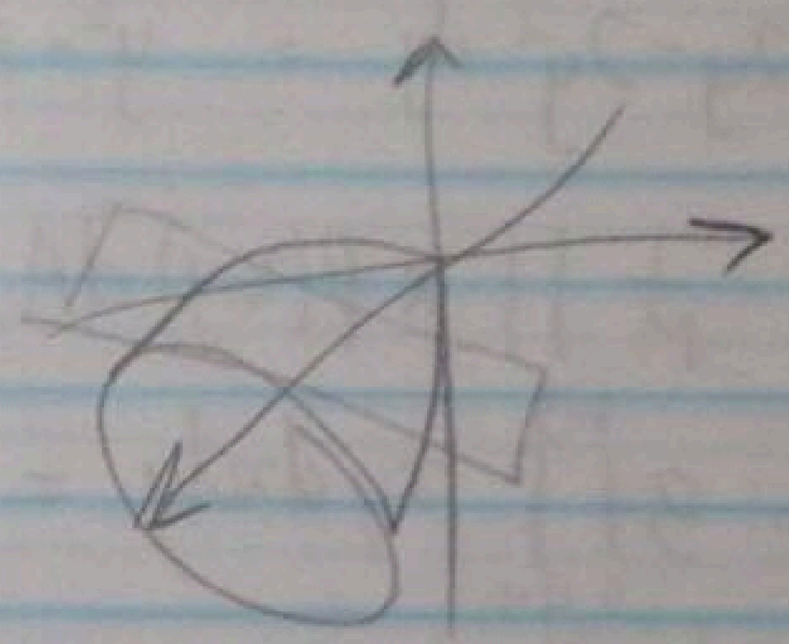
$= \displaystyle\frac{1}{2}\int_0^1 -\frac{1}{2}\cdot\frac{1}{3}(2-2x-2y)^3\Big|_0^{1-x}$

$= \displaystyle\frac{1}{12}\int_0^1 (2-2x)^3\,dx$

$= \displaystyle-\frac{1}{12}\left[-\frac{1}{2}(2-2x)^4\frac{1}{4}\right]_0^1$

$= \displaystyle\frac{1}{12}\cdot\frac{1}{8}\left[2^4\right]$

$= \displaystyle\frac{1}{32\cdot 3}\cdot 16 = \frac{1}{6}$



② $\displaystyle\iiint x\,dV$

$0 \le \theta \le 2\pi$

$0 \le r \le \frac{\sqrt{x}}{2}$

$4r^2 \le x \le 4$

$x = 4(y^2+z^2)$

$x = 4$

$x = 4r^2$

$r = \frac{\sqrt{x}}{4}$

$\displaystyle\int_0^{2\pi}\int_0^1\int_{4r^2}^4 x\,r\,dx\,dr\,d\theta$

$= \displaystyle 2\pi\int_0^1 \frac{1}{2}r\,x^2\Big|_{4r^2}^4\,dr$

$= \displaystyle \pi\int_0^1 r\left((4)^2-(4r^2)^2\right)$

$= \displaystyle \pi\int_0^1 r(16-16r^4)$

$= \displaystyle 16\pi\int_0^1 r - r^5\,dr$

$= \displaystyle 16\pi\left(\frac{r^2}{2}-\frac{r^6}{6}\right)\Big|_0^1 = 16\pi\left(\frac{1}{2}-\frac{1}{6}\right) = 16\pi\left(\frac{3}{6}-\frac{1}{6}\right) = \frac{16\pi\cdot 2}{6}$

$= \displaystyle\frac{32}{6}\pi = \boxed{\frac{16}{3}\pi}$

Problem 4: (4)

Find the iterated integral

$$\int_0^R \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \sin(x^2+y^2)\,dy\,dx.$$

Hint: Convert the iterated integral into a proper double integral and then evaluate the double integral.

$x^2+y^2=r^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$$\int_0^K \int_{\sqrt{K^2-r^2\cos^2\theta}}^{\sqrt{K^2-r\sin\theta}} r\sin(r^2)\,dy\,dx$$

$$\int_0^{2\pi}\int_0^K r\sin(r^2)\,dr\,d\theta$$

$$2r^2\sin(2r) - 4r^2\sin(2r)\Big|_0^{r=K}$$

$$\int_0^{2\pi} 2K^2\sin(2K) - 4K^2\sin(2K)\,d\theta$$

$$4\pi K^2\sin(2K) - 8\pi K^2\sin(2K)$$

$$= \cancel{-4\pi K^2\sin(2K)}$$

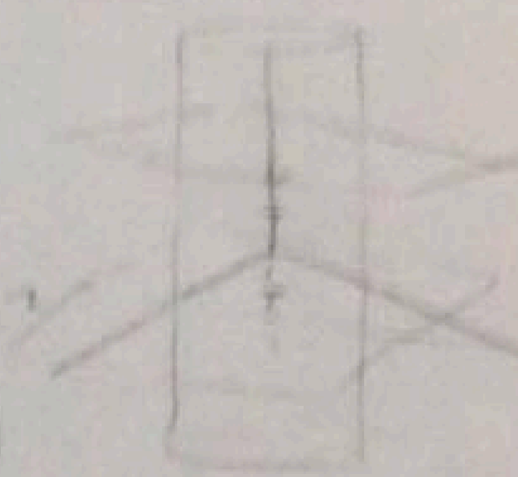
Problem 5: (4)

Estimate the following integral

$$\iiint_E e^{\sin x \sin y \sin z}\,dV,$$

where E is the solid bounded by the cylinder $x^2+y^2=R^2$, the plane $z=-2$ and the plane $z=2$.

$x^2+y^2=r^2=R^2$



$$\int_0^{2\pi}\int_0^R\int_{-2}^{2} e^{(\sin(r\cos\theta))(\cos(r\sin\theta))\sin z}\,dz\,dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\frac{e^{(\sin(r\cos\theta))(\cos(r\sin\theta))(\sin(z))}}{\sin(z)}\Bigg|_{z=-2}^{z=2}$$

$-1 \le \sin x \cos y \sin z \le +1$

$\Rightarrow -1 \le e^{\ldots} \le 1$

$\int_E e^{-1} = e^{-1}\,\mathrm{vol}(E) \qquad e^{-1}\mathrm{vol}(E) \le \int_E f \le e^{+1}\,\mathrm{vol}(E)$

$\int_E e^{+1} = e^{+1}\,\mathrm{vol}(E)$

1

[redacted]

1	2
2	4
3	2
4	2.5
5	0
T	10.5

MATH 32B Midterm I, Winter 2012

Name: [redacted] TA's Name and Section Number: 3B Krause

Problem 1. (4)

Find the triple integral $\int\int\int_E z\,dxdydz$. Here E is the solid bounded by the four planes $x=0$, $y=0$, $z=0$ and $2x+2y+z=2$.

$z\Big|_0^{2-2x-2y}$ $\quad z = 2-2x+2y$

$2x+2y=2$

$y = 1-x$

$y\Big|_0^{1-x}$

$x\Big|_0^{1}$

$$\boxed{\int_0^1\int_0^{1-x}\int_0^{2-2x-2y} z\,dz\,dy\,dx =} \quad 1/2$$

$z\,dy\,dz\,dx$

$(2-2x)(2-2x)$

$4-8x+4x^2$

$2-2x=K$

$$\frac{1}{2}z^2\Big|_0^{2-2x-2y} = \int_0^{1-x}\frac{1}{2}(2-2x-2y)^2\,dy = \frac{1}{2}(K-2y)(K-2y) \quad K^2-4yK+4y^2$$

$$= \int_0^{y=1-x}\frac{K^2}{2}-2yK+4y^2\,dy = \frac{K^2}{2}y - y^2K + \frac{4}{3}y^3\Big|^1$$

$$\int^1 (\ldots x+2x^2)(1-x) - (1-x)^2(2-x) + \frac{4}{3}(1-x)^3$$

$$(4x-4x^2)+2(1-x)^2-(1-2x+x^2)(2-x)+\tfrac{4}{3}(1-x)^3$$

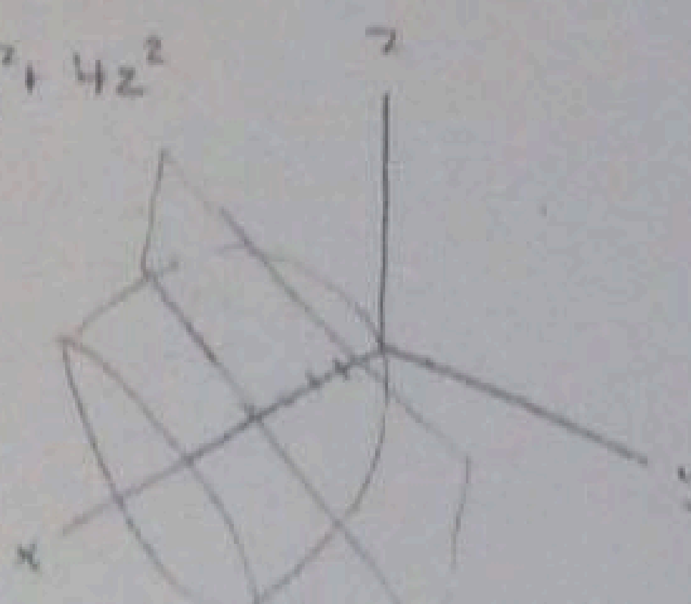
Problem 2. (4)

Find the triple integral

$$\iiint_E x\,dV,$$

where E is the region bounded by the paraboloid $x = 4(y^2 + z^2)$ and the plane $x = 4$.

$x = 4y^2 + 4z^2$



$\theta \Big|_0^{2\pi} \quad x \Big|_0^{4} \quad r \Big|_0^{\frac{\sqrt{x}}{2}}$

$$\int_0^{2\pi}\int_0^{4}\int_0^{\frac{\sqrt{x}}{2}} x\,r\,dr\,dx\,d\theta$$

$y^2 + z^2 = r^2$

$x = 4r^2$

$r = \sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{2}$

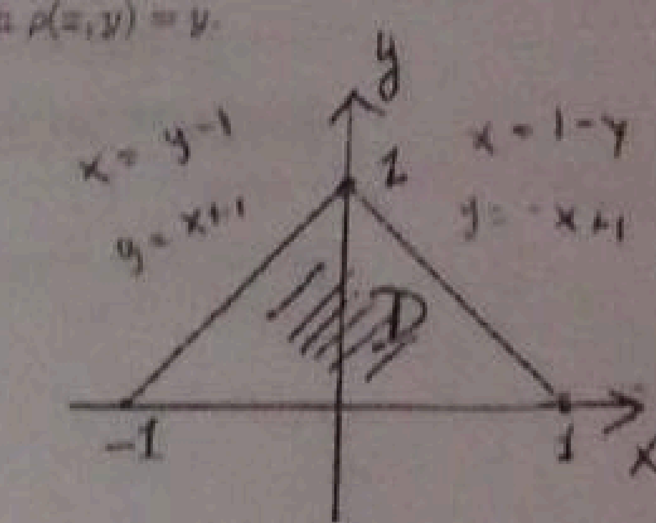
$\frac{1}{2} r^2 x \Big|_0^{r=\frac{\sqrt{x}}{2}} = \frac{1}{2}\left(\frac{x}{4}\right)x = \frac{x^2}{8}$

$\int_0^4 \frac{x^2}{8}\,dx \quad \frac{x^3}{24}\Big|_0^4 = \frac{64}{24} = \frac{32}{12} = \frac{16}{6} = \frac{8}{3}$

$\int_0^{2\pi} 8/3\,d\theta = 8/3\,\theta \Big|_0^{2\pi} = \boxed{\frac{16\pi}{3}}$

Problem 3. (4)

Compute the center of mass of D. Here the density fuction $\rho(x,y) = y$.



$m = \iint_D \rho\,dx\,dy$

$= \int_0^1\int_{y-1}^{1-y} y\,dx\,dy =$

$yx \Big|_{y-1}^{1-y} = y(1-y) - y(y-1)$

$= y - y^2 - y^2 + y = 2y - 2y^2$

$\int_0^1 (2y - 2y^2)\,dy = y^2 - \frac{2}{3}y^3 \Big|_0^1 = 1 - \frac{2}{3}(1) = \frac{1}{3} = m$

$\bar{x} = \frac{1}{m}\int_0^1\int_{y-1}^{1-y} xy\,dx\,dy = \frac{2}{m}\int_0^1\int_{0}^{1} xy\,dx\,dy \quad \frac{1}{2}x^2 y \Big|_{y-1}^{1-y}$

$\frac{1}{2}x^2 y \Big|_0^{1-y} = \frac{1}{2}(1-y)^2 y$

$\frac{1}{2}(y^2 - 2y + 1)y = \int_0^1 \frac{y^3}{2} - y^2 + \frac{y}{2}$

$\frac{y^3}{3} + \frac{y^2}{4}\Big|^1 = \frac{1}{8} - \frac{1}{3} + \frac{1}{4} = \frac{3}{24} - \frac{8}{24} + \frac{6}{24} = \frac{1}{24}(6) = \frac{1}{4}\ (\bar{x})$

$y^2 x \Big|_0^{1-y} = y^2(1-y) = \int_0^1 y^2 - y^3\,dy = \frac{1}{3}y^3 - \frac{1}{4}y^4 \Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} \quad \frac{1}{4}(6) = 1\frac{1}{2}$

$= \bar{y}$

$\boxed{\text{center} = (\cancel{1/4},\ 1\tfrac{1}{2})}$