32B Lect. 1

Midterm 1

January 31

First Name: [redacted]

ID# [redacted]

Last Name [redacted]

Section: 1F

$$= \begin{cases} 1a & \text{Tuesday with Eric Auld} \\ 1b & \text{Thursday with Eric Auld} \\ 1c & \text{Tuesday with Kyung Ha} \\ 1d & \text{Thursday with Kyung Ha} \\ 1e & \text{Tuesday with Khang Huynh} \\ 1f & \text{Thursday with Khang Huynh} \end{cases}$$

Rules.

- There are **FOUR** problems; ten points per problem.
- There is an extra page at the end. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	Σ
10	4	10	~~6~~ 5	29

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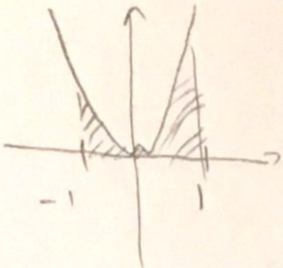
(1) Find the center of mass for a homogeneous planar body occupying the region where

$$-1 \le x \le 1 \quad \text{and} \quad 0 \le y \le x^4$$

$$\text{CoM}_x = \left(\frac{1}{A}\iint x \, dA \, , \, \frac{1}{A}\iint y \, dA\right)$$



$$A = \int_{-1}^{1}\int_{0}^{x^4} 1 \, dy\, dx$$

$$A = \int_{-1}^{1} x^4 \, dx$$

$$A = \frac{1}{5}x^5 \Big|_{-1}^{1}$$

$$A = \frac{1}{5}(1)^5 - \frac{1}{5}(-1)^5$$

$$= \frac{2}{5}$$

$$\text{com}_x = \frac{5}{2} \cdot \int_{-1}^{1}\int_{0}^{x^4} x \, dy\, dx$$

$$= \frac{5}{2} \cdot \int_{-1}^{1} x^5 \, dx$$

$$= \frac{5}{2} \cdot \left(\frac{1}{6}x^6 \Big|_{-1}^{1}\right)$$

$$= \frac{5}{2} \cdot \left(\frac{1}{6}(1) - \frac{1}{6}(1)\right)$$

$$= 0$$

$$\text{com}_y = \frac{5}{2} \cdot \int_{-1}^{1}\int_{0}^{x^4} y \, dy\, dx$$

$$= \frac{5}{2} \cdot \int_{-1}^{1} \left(\frac{1}{2}y^2\right)\Big|_{0}^{x^4} dx$$

$$= \frac{5}{2}\int_{-1}^{1} \frac{1}{2}(x^4)^2 \, dx$$

$$= \frac{5}{2}\int_{-1}^{1} \frac{1}{2}x^8 \, dx$$

$$= \frac{5}{2} \cdot \left(\frac{1}{18}x^9 \Big|_{-1}^{1}\right)$$

$$= \frac{5}{2} \cdot \left(\frac{1}{18}(1) - \frac{1}{18}(-1)\right)$$

$$= \frac{5}{2} \cdot \frac{1}{9}$$

$$= \frac{5}{18}$$

$$\boxed{\text{center of mass} = \left(0, \frac{5}{18}\right)}$$

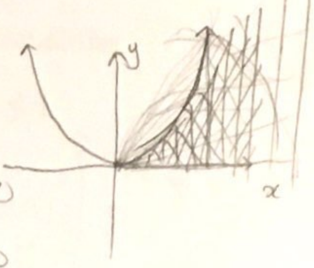
(2) Evaluate the integral

$\sqrt{y} = x$ $\quad$ $y = r\sin\theta$

$y = x^2$ $\quad$ $x^2 + y^2 = r^2$

$$\int_0^\infty \int_{\sqrt{y}}^\infty \frac{y}{(x^2+y^2)^2}\,dx\,dy$$

by converting to polar coordinates.



$\sqrt{y} \le x \le \infty$ $\quad$ $y \le x^2 \le \infty$

$0 \le y \le \infty$ $\quad$ $0 \le y \le \infty$

$= \quad \sqrt{r\sin\theta} \le r\cos\theta \le \infty$

$0 \le r\sin\theta \le \infty$ $\quad$ $r\sin\theta \le r^2\cos^2\theta \le \infty$

$0 \le \sin\theta \le \infty$ $\quad$ $\frac{r\sin\theta}{r^2\cos^2\theta} \le 1 \le \infty$ $\sim$

$0 \le \theta \le 2\pi$ $\quad$ $\tan\theta\sec\theta \le r \le \infty$

$0 \le r \le \infty$

$\int_0^\infty \int_0^{2\pi} \frac{r\sin\theta}{(r^2)^2}\, r\, d\theta\, dr - \int$

$\int_0^\infty \int_0^{2\pi} \frac{\sin\theta}{r^2}\, d\theta\, dr$

$\int_0^\infty \frac{1}{r^2}\left(-\cos\theta \Big|_0^{2\pi}\right) dr$

$\int_0^\infty \frac{1}{r^2}\left(-(1)-(-1)\right) dr$

$\boxed{0}$

$= \int_0^{2\pi} \int_{\tan\theta\sec\theta}^{\infty} \frac{\sin\theta}{r^2}\, dr\, d\theta$

$= \int_0^{2\pi} -\frac{1}{3}\sin\theta\, r^{-3} \Big|_{\tan\theta\sec\theta}^{\infty} d\theta$

$= -\frac{1}{3} \cdot \int_0^{2\pi} \frac{\sin\theta}{r^3} \Big|_{\tan\theta\sec\theta}^{\infty} d\theta$

$= -\frac{1}{3}\int_0^{2\pi} \tan^2\theta\, d\theta$

$=$

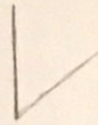
4

$x^2 \le 1$

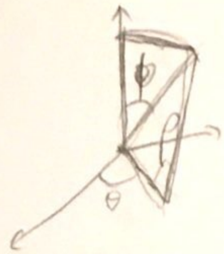
$x \le 1$

$\varphi^2 \le 1$

$y \le$

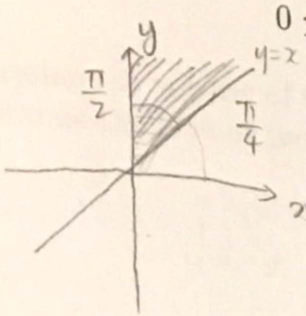


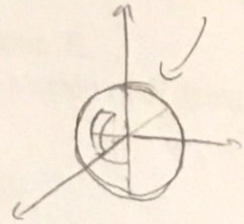
$\frac{y}{x}$

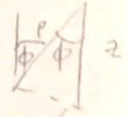


(3) Determine the volume of the region defined by the following inequalities

$$0 \le x \le y \quad \text{and} \quad x^2 + y^2 + z^2 \le 1$$







$0 \le x \le y$

$x^2+y^2+z^2 \le 1$

$x^2+y^2+z^2 \le 1$

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi \le 1$

$0 \le \rho\sin\phi\cos\theta \le \rho\sin\phi\sin\theta$

$\rho^2\sin^2\phi + \rho^2\cos^2\phi \le 1$

$0 \le \cos\theta \le \sin\theta$

$\rho^2 \le 1$

$0 \le \cot\theta \le 1$

$0 \le \rho \le 1$

$0 \le \phi \le \pi$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$V = \int_0^{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^1 1\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$V = \int_0^{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tfrac{1}{3}\rho^3\Big|_0^1 \sin\phi\ d\theta\ d\phi$$

$$V = \int_0^{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tfrac{1}{3}\sin\phi\ d\theta\ d\phi$$

$$V = \int_0^{\pi} \tfrac{\pi}{12}\sin\phi\ d\phi$$

$$V = \tfrac{\pi}{12}\cdot\left(-\cos\phi\Big|_0^{\pi}\right) d\phi$$

$$V = \tfrac{\pi}{12}\left(-(-1)-(-(1))\right)$$

$$V = \tfrac{\pi}{12}(1+1)$$

$$\boxed{V = \frac{\pi}{6}}$$

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(4) Consider the region $\mathcal{R}$ defined by the inequalities

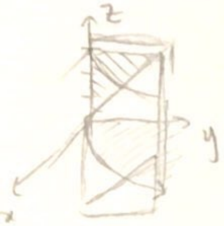
$$0 \le x^2 \le y \le z \le 3$$

(a) Determine the volume of the region $\mathcal{R}$.
(b) Determine the area of the cross-section of $\mathcal{R}$ lying in the plane $y = 1$.

4

3
9
27
81
243

a) $0 \le x^2$
$x^2 \le y$
$y \le z$
$z \le 3$



$V = \iiint_R 1\, dV$

$V = \iiint 1[0 \le x^2][x^2 \le y][y \le z][z \le 3]\, dV$

$0 \le x^2 \le y \Rightarrow 0 \le x \le \sqrt{y}$

$V = \iint [y \le z][z \le 3] \int_{-\sqrt{y}}^{\sqrt{y}} 1\, dx\, dy\, dz$

$\frac{4}{15}$

$V = \iint [y \ge 0][y \le z][z \le 3] \int_0^{\sqrt{y}} 1\, dx\, dy\, dz$

$y \quad \frac{5}{2}$

$V = \int [z \le 3] \int_0^z \int_0^{\sqrt{y}} 1\, dx\, dy\, dz$

$V = \int_0^3 \int_0^z \int_0^{\sqrt{y}} 1\, dx\, dy\, dz$

$(y)^{\frac{1}{2}}$

$\frac{2}{3} y^{\frac{3}{2}}$

$V = \int_0^3 \int_0^z \sqrt{y}\, dy\, dz$

$V = \int_0^3 \frac{2}{3} y^{\frac{3}{2}} \Big|_0^z\, dz$

$V = \int_0^3 \frac{2}{3} z^{\frac{3}{2}}\, dz$

$V = \frac{4}{15} z^{\frac{5}{2}} \Big|_0^3$

$V = \frac{4}{15} (3)^{\frac{5}{2}}$

$V = \frac{4}{15} \sqrt{243}$

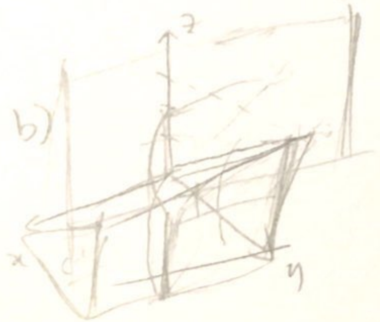
20 9 2
32 227 3
3
81 4
3
243 5

$3\sqrt{243}$

$\begin{array}{r} 81 \\ 3\overline{)243} \end{array}$

3
$9\sqrt{3}$

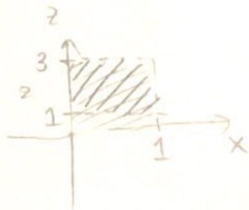
b)



$0 \le x^2 \le 1 \le z \le 3$

$0 \le x^2 \le 1 \quad 0 \le x \le 1$

$1 \le z \le 3$



$= \text{box } [0,1] \times [1,3]$

$\boxed{A = 2}$ 4

integral