Math 32B - Lecture 4
Winter 2021
Midterm 2
Due 2/25/2021 before 10am

Sign and submit the following honor statement:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signed:

Print name:

This exam contains 5 pages (including this cover page) and 4 problems. There are a total of 30 points available.

- Attempt all questions.
- Solutions must be uploaded to Gradescope before 10am Pacific Time on February 25^{th}.
  - Include extra pages as you need them.
  - You may complete the problems on a printout of this exam, blank paper, or a tablet/iPad.
  - If you handwrite your solutions, please make sure your scan is clearly legible.
  - Solutions should be written clearly, in full English sentences, defining all variables, showing all working, and giving units where appropriate.
- The work submitted must be entirely your own: you may not collaborate or work with anyone else to complete the exam.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1. (6 points) Let $\mathscr{C}$ be the curve with parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $-4\pi \leq t \leq 4\pi$. Find the value of the constant C that gives the identity

$$\text{length}(\mathscr{C}) = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r},$$

where the vector field

$$\mathbf{F}(x, y, z) = \big\langle -y \, , \, x \, , \, C \big\rangle.$$

Solution:

We first compute

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

and hence

$$\|\mathbf{r}'(t)\| = \sqrt{2}.$$

We then have

$$\text{length}(\mathscr{C}) = \int_{-4\pi}^{4\pi} \sqrt{2} \, dt = 8\sqrt{2}\pi.$$

Second, we compute

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{-4\pi}^{4\pi} \langle -\sin t, \cos t, C \rangle \cdot \langle -\sin t, \cos t, 1 \rangle \, dt = \int_{-4\pi}^{4\pi} C + 1 \, dt = 8\pi(C + 1).$$

As a consequence, we must take

$$C = \sqrt{2} - 1.$$

2. (7 points) A solid $\mathscr{W}$ occupies the region $x^2+y^2+z^2 \leq 25$ and $z \leq -\sqrt{x^2+y^2}$, where distance is measured in cm.

The solid has mass density

$$f(x,y,z) = -\frac{z}{\sqrt{x^2+y^2+z^2}} \quad \text{g cm}^{-3}.$$

Use spherical coordinates to compute the total mass of the solid.

Solution:

We observe that our region can be written in spherical coordinates as

$$\mathscr{W} = \left\{ 0 \leq \rho \leq 5\,,\ \tfrac{3\pi}{4} \leq \phi \leq \pi\,,\ 0 \leq \theta < 2\pi \right\}.$$

We then compute the mass

$$\begin{aligned} M &= \int_0^{2\pi} \int_{\frac{3\pi}{4}}^{\pi} \int_0^5 \big(-\cos\phi\big)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= -\int_0^{2\pi} \int_{\frac{3\pi}{4}}^{\pi} \int_0^5 \tfrac{1}{2}\rho^2 \sin(2\phi) \, d\rho \, d\phi \, d\theta \\ &= -\int_0^{2\pi} \int_{\frac{3\pi}{4}}^{\pi} \tfrac{125}{6} \sin(2\phi) \, d\phi \, d\theta \\ &= \int_0^{2\pi} \tfrac{125}{12} \, d\theta \\ &= \tfrac{125}{6}\pi \quad \text{g} \end{aligned}$$

3. (8 points) Let $\mathscr{D}$ be the region bounded between $(x-1)^2+y^2=1$ and $(x-2)^2+y^2=4$. Use the change of variables $(x,y)=(r+r\cos\theta, r\sin\theta)$ to evaluate

$$\iint_{\mathscr{D}} \frac{x^2+y^2}{x}\, dA.$$

Solution: Using the variables $(x,y)=(r+r\cos\theta, r\sin\theta)$, our region $\mathscr{D}$ is given by $1 \leq r \leq 2$ and $0 \leq \theta < 2\pi$. Next, we compute the Jacobian

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} 1+\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r + r\cos\theta.$$

Observe that this is non-negative, so

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r + r\cos\theta.$$

Next, we compute that

$$x^2+y^2 = 2r^2 + 2r^2\cos\theta,$$

and hence

$$\frac{x^2+y^2}{x} = 2r.$$

Consequently,

$$\iint_{\mathscr{D}} \frac{x^2+y^2}{x}\, dA = \int_0^{2\pi}\int_1^2 2r^2(1+\cos\theta)\, dr\, d\theta = \frac{28}{3}\pi.$$

4. (9 points) Let S be the part of the cylinder $y^2 + z^2 = 1$ bounded between $z = 0$, $z = 1 + x$ and $z = 1 - x$, oriented with the downward pointing normal.

   Find the flux of the vector field

$$\mathbf{F}(x, y, z) = \langle 0\,,\, y,\, z\rangle$$

   across S.

Solution: We parameterize S using

$$G(x, \theta) = (x\,,\, \cos\theta\,,\, \sin\theta),$$

for $0 \le \theta \le \pi$ and $\sin\theta - 1 \le x \le 1 - \sin\theta$.

We then compute

$$\mathbf{T}_x = \langle 1\,,\, 0\,,\, 0\rangle \quad \text{and} \quad \mathbf{T}_\theta = \langle 0\,,\, -\sin\theta\,,\, \cos\theta\rangle.$$

The corresponding normal is

$$\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_\theta = \langle 0\,,\, -\cos\theta\,,\, -\sin\theta\rangle,$$

which we note is correctly oriented.

We then compute the flux to be

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^\pi \int_{\sin\theta - 1}^{1 - \sin\theta} \langle 0\,,\, \cos\theta\,,\, \sin\theta\rangle \cdot \langle 0\,,\, -\cos\theta\,,\, -\sin\theta\rangle\, dx\, d\theta \\ &= \int_0^\pi 2\sin\theta - 2\, d\theta \\ &= 4 - 2\pi.\end{aligned}$$