**Math 32B - Lecture 4**
**Winter 2021**
**Midterm 1**
**Due 1/28/2021 before 10am**

---

**Sign and submit the following honor statement:**

> *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

**Signed:**

**Print name:**

---

This exam contains 6 pages (including this cover page) and 4 problems. There are a total of 30 points available.

- Attempt all questions.
- Solutions must be uploaded to Gradescope before 10am Pacific Time on January 28$^{\text{th}}$.
  - Include extra pages as you need them.
  - You may complete the problems on a printout of this exam, blank paper, or a tablet/iPad.
  - If you handwrite your solutions, please make sure your scan is clearly legible.
  - Solutions should be written clearly, in full English sentences, defining all variables, showing all working, and giving units where appropriate.
- The work submitted must be entirely your own: you may not collaborate or work with anyone else to complete the exam.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1. (4 points) Let $\mathscr{W} = [0,1] \times [1,2] \times [2,3]$. Find

$$\iiint_{\mathscr{W}} xyz\, dV$$

**Solution:** Applying Fubini's Theorem, we have

$$\iiint_{\mathscr{W}} xyz\, dV = \int_2^3 \int_1^2 \int_0^1 xyz\, dx dy dz = \int_2^3 \int_1^2 \tfrac{1}{2} yz\, dy dz = \int_2^3 \tfrac{3}{4} z\, dz = \tfrac{15}{8}.$$

2. (8 points) Evaluate

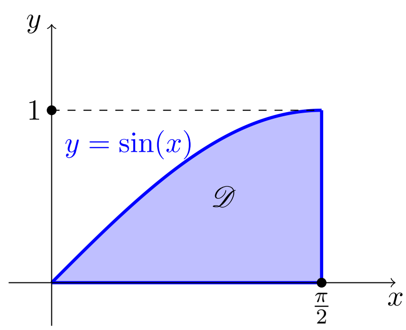
$$\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} e^{\cos(x)}\,dx\,dy.$$

You should assume that $\sin^{-1}(y)$ has range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(*Hint:* *At some point in your solution, it might be useful to use the substitution* $u = \cos x$.)

**Solution:** We take $\mathscr{D}$ to be the region where

$$0 \leq y \leq 1 \quad \text{and} \quad \sin^{-1}(y) \leq x \leq \tfrac{\pi}{2}$$




We may write this as a vertically simply region

$$0 \leq x \leq \tfrac{\pi}{2} \quad \text{and} \quad 0 \leq y \leq \sin(x).$$

By Fubini's Theorem, we then have

$$\begin{aligned}\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} e^{\cos(x)}\,dx\,dy &= \iint_{\mathscr{D}} e^{\cos(x)}\,dA \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\sin(x)} e^{\cos(x)}\,dy\,dx \\ &= \int_0^{\frac{\pi}{2}} \sin(x) e^{\cos(x)}\,dx\end{aligned}$$

Introducing $u = \cos(x)$ we have

$$\int_0^{\frac{\pi}{2}} \sin(x) e^{\cos(x)}\,dx = \int_0^1 e^u\,du = e - 1.$$

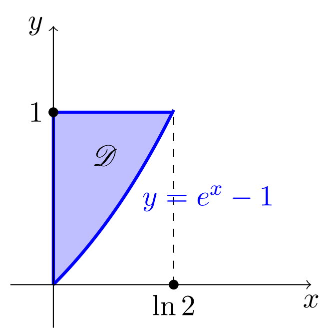
3. (10 points)

   (a) Let $\mathscr{D}$ be the region in the $(x, y)$-plane bounded by $x = 0$, $y = e^x - 1$, and $y = 1$. Use a double integral to compute the area of $\mathscr{D}$.

   (b) Let $\mathscr{W}$ be the 3$d$ region above $\mathscr{D}$ and below the surface $z = 1 - y$. Use a triple integral to compute the volume of $\mathscr{W}$.

**Solution:**

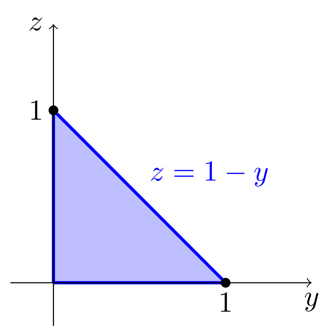
(a) We first sketch $\mathscr{D}$:




We then compute the area

$$\text{Area}(\mathscr{D}) = \iint_{\mathscr{D}} dA = \int_0^{\ln 2} \int_{e^x-1}^{1} dydx = \int_0^{\ln 2} 2 - e^x \, dx = 2\ln 2 - 1.$$

(b) The projection of $\mathscr{W}$ onto the $(x, y)$-plane is simply $\mathscr{D}$. The projection onto the $(y, z)$-plane is easily seen to be:



We then compute

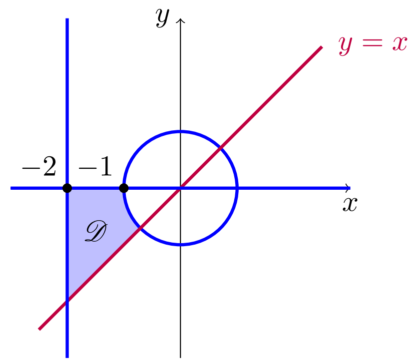
$$\begin{aligned}
\text{Volume}(\mathscr{W}) &= \iiint_{\mathscr{W}} dV \\
&= \int_0^{\ln 2} \int_{e^x-1}^{1} \int_0^{1-y} dz\,dy\,dx \\
&= \int_0^{\ln 2} \int_{e^x-1}^{1} 1 - y\,dy\,dx \\
&= \int_0^{\ln 2} 2 - 2e^x + \tfrac{1}{2}e^{2x}\,dx \\
&= 2\ln 2 - \tfrac{5}{4}
\end{aligned}$$

4. (8 points) Let $\mathscr{D}$ be the region where $-2 \le x \le y \le 0$ and $x^2 + y^2 \ge 1$. Evaluate

$$\iint_{\mathscr{D}} (x^2 + y^2)^{-\frac{3}{2}}\, dA.$$

**Solution:**

We start by sketching our domain $\mathscr{D}$:




In polar coordinates, we may write $\mathscr{D}$ as the region where

$$1 \le r \le -\frac{2}{\cos\theta} \quad \text{and} \quad \pi \le \theta \le \tfrac{5\pi}{4}.$$

We then compute

$$\iint_{\mathscr{D}} (x^2 + y^2)^{-\frac{3}{2}}\, dA = \int_{\pi}^{\frac{5\pi}{4}} \int_{1}^{-\frac{2}{\cos\theta}} \tfrac{1}{r^3}\, r\, dr\, d\theta = \int_{\pi}^{\frac{5\pi}{4}} 1 + \tfrac{1}{2}\cos\theta\, d\theta = \frac{\pi - \sqrt{2}}{4}.$$