Math 32B - Lecture 1
Fall 2021
Midterm 2
11/17/2021

Name: ______________________________

UID: ______________________________

Time Limit: 50 mins

Version $\uparrow$

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 25 points available.

Check to see if any pages are missing. Enter your name and UID at the top of this page.

You are allowed one handwritten side of US letter paper with notes. You may **not** use any other notes, books, calculators, or other assistance.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

1. (4 points) Let $\mathscr{C}$ be the straight line from $(3, 1)$ to $(-2, 10)$. Find

$$\int_{\mathscr{C}} e^y \, dx.$$

Solution: Let us parameterize $\mathscr{C}$ using

$$\mathbf{r}(t) = t\langle -2, 10\rangle + (1-t)\langle 3, 1\rangle = \langle 3-5t, 1+9t\rangle \quad \text{for} \quad 0 \le t \le 1.$$

We then compute that

$$\mathbf{r}'(t) = \langle -5, 9\rangle$$

and hence

$$\int_{\mathscr{C}} e^y \, dx = \int_0^1 e^{1+9t}\,(-5)\, dt = -\tfrac{5}{9}\big(e^{10} - e\big).$$

2. (6 points) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4$ where $x \leq 1$.

Solution: We parameterize the sphere using

$$G(\theta, \phi) = \big(2\cos\phi, 2\cos\theta\sin\phi, 2\sin\theta\sin\phi\big),$$

where $0 \leq \theta < 2\pi$ and $\frac{\pi}{3} \leq \phi \leq \pi$. From class, we know that

$$\|\mathbf{N}\| = 4\sin\phi,$$

and hence

$$\text{Area}(\mathcal{S}) = \iint_{\mathcal{S}} dS = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi} 4\sin\phi\, d\phi\, d\theta = 12\pi.$$

3. (8 points) Let $\mathcal{S}$ be the part of $z = 1-(x-y)^2$ in the region where $x \leq 0$, $y \geq 0$, and $y \leq x+1$, oriented with downwards pointing unit normal. Find

$$\iint_{\mathcal{S}} \langle z, z, z\rangle \cdot d\mathbf{S}.$$

Solution: We parameterize the surface $\mathcal{S}$ using

$$G(y, x) = \big(x, y, 1-(x-y)^2\big),$$

for $-1 \leq x \leq 0$ and $0 \leq y \leq x+1$. We compute that

$$\mathbf{T}_y = \frac{\partial G}{\partial y} = \big\langle 0, 1, 2(x-y)\big\rangle \quad \text{and} \quad \mathbf{T}_x = \frac{\partial G}{\partial x} = \big\langle 1, 0, -2(x-y)\big\rangle,$$

and hence

$$\mathbf{N} = \mathbf{T}_y \times \mathbf{T}_x = \big\langle -2(x-y), 2(x-y), -1\big\rangle.$$

We then have

$$\begin{aligned}
\iint_{\mathcal{S}} \langle z, z, z\rangle \, d\mathbf{S} &= \int_{-1}^{0}\int_{0}^{x+1} \big\langle 1-(x-y)^2, 1-(x-y)^2, 1-(x-y)^2\big\rangle \cdot \big\langle 2-(x-y), 2(x-y), -1\big\rangle \, dy\, dx \\
&= \int_{-1}^{0}\int_{0}^{x+1} (x-y)^2 - 1 \, dy dx \\
&= \int_{-1}^{0} \Big[-\tfrac{1}{3}(x-y)^3 - y\Big]_{y=0}^{y=x+1} dx \\
&= \int_{-1}^{0} -\tfrac{1}{3}x^3 - x - \tfrac{2}{3}\, dx \\
&= \Big[\tfrac{1}{12}x^4 - \tfrac{1}{2}x^2 - \tfrac{2}{3}x\Big]_{x=-1}^{0} \\
&= -\tfrac{1}{4}
\end{aligned}$$

4. (7 points)

   (a) Show that the vector field
   $$\mathbf{F}(x,y,z) = \big\langle z, x, x+y \big\rangle$$
   is not conservative on $\mathbb{R}^3$.

   (b) Find a potential function for the conservative vector field
   $$\mathbf{G}(x,y,z) = \big\langle x+y, x, z \big\rangle.$$

   (c) Let $\mathscr{C}$ be a curve in the plane $z = x+y$ from the point $(1,0,1)$ to the point $(0,-1,-1)$. Taking $\mathbf{F}$ as in part (a), find
   $$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r}.$$

Solution:

(a) We compute that
$$\operatorname{curl} \mathbf{F} = \langle 1, 0, 1 \rangle \neq 0,$$
so, as shown in class, this is not conservative.

(b) If g is a potential function for $\mathbf{G}$ then comparing x-components we have
$$\frac{\partial g}{\partial x} = x + y,$$
so
$$g(x,y,z) = \tfrac{1}{2}x^2 + xy + C(y,z)$$
for some function $C(y,z)$. Next, we compare y-components to find that
$$\frac{\partial C}{\partial y} = 0,$$
and hence $C = C(z)$. Finally, we compare z-components to find that
$$C'(z) = z.$$
As a consequence,
$$g(x,y,z) = \tfrac{1}{2}x^2 + xy + \tfrac{1}{2}z^2,$$
is a potential function for $\mathbf{G}$.

(c) Observe that for all (x,y,z) in the plane $z = x+y$ we have $\mathbf{F}(x,y,z) = \mathbf{G}(x,y,z)$. In particular, we have
$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathscr{C}} \mathbf{G} \cdot d\mathbf{r}.$$
Taking g to the potential for $\mathbf{G}$ found in part (b), we may then use the Fundamental Theorem for Vector Line Integrals to compute that
$$\int_{\mathscr{C}} \mathbf{G} \cdot d\mathbf{r} = g(0,-1,-1) - g(1,0,1) = \tfrac{1}{2} - 1 = -\tfrac{1}{2}.$$