Math 32B - Lecture 1
Fall 2021
Final Exam
12/10/2021

Name: ______________________

UID: ______________________

Time Limit: 180 mins

Version $\leftarrow$

This exam contains 19 pages (including this cover page) and 8 problems. There are a total of 70 points available.

Check to see if any pages are missing. Enter your name and UID at the top of this page.

You are allowed one two-sided handwritten page of US letter paper with notes. You may **not** use any other notes, books, calculators, or other assistance.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

1. (6 points) Write the iterated integral

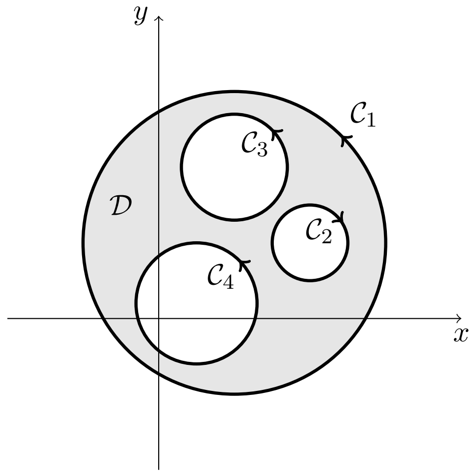
$$\int_0^1 \int_0^z \int_y^z e^{\cos(x)\sin(y)z}\, dx\, dy\, dz$$

in the order $dz\, dy\, dx$.

Solution: Let $\mathcal{W} = \{0 \leq y \leq x \leq z \leq 1\}$. Then,

$$\int_0^1 \int_0^z \int_y^z e^{\cos(x)\sin(y)z}\, dx\, dy\, dz = \iiint_{\mathcal{W}} e^{\cos(x)\sin(y)z}\, dV = \int_0^1 \int_0^x \int_x^1 e^{\cos(x)\sin(y)z}\, dz\, dy\, dx$$

2. (6 points) Consider the region $\mathcal{D}$ bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:



Let $\mathbf{F}$ be a smooth vector field satisfying $\text{curl}_z\, \mathbf{F} = 0$ in $\mathcal{D}$ and

$$\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 20, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 21, \qquad \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = 22.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$.

Solution: By Green's Theorem, we have

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \text{curl}_z\, \mathbf{F}\, dA = 0.$$

By considering the orientation, we see that

$$\begin{aligned}\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} &= \oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} - \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} - \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - 23,\end{aligned}$$

and hence

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 23.$$

3. (8 points) Let $\mathcal{D}$ be the parallelogram with vertices $(0,0)$, $(2,1)$, $(3,3)$, $(1,2)$. Using a suitable change of variables, compute

$$\iint_{\mathcal{D}} y\,dA.$$

Solution: Let us write

$$x = 2u + v \quad \text{and} \quad y = u + 2v,$$

so that $\mathcal{D} = \{0 \le u \le 1, 0 \le v \le 1\}$. We then compute that the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3.$$

As a consequence,

$$\iint_{\mathcal{D}} y\,dA = \int_0^1 \int_0^1 3(u+2v)\,du\,dv = \tfrac{9}{2}.$$

4. (8 points) Let $\mathcal{C}$ be the part of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $-2\pi \leq t \leq 2\pi$. Find

$$\int_{\mathcal{C}} \frac{z}{x^2 + y^2 + z^2}\, ds.$$

Solution: We compute that

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

and hence

$$\|\mathbf{r}'(t)\| = \sqrt{2}.$$

As a consequence,

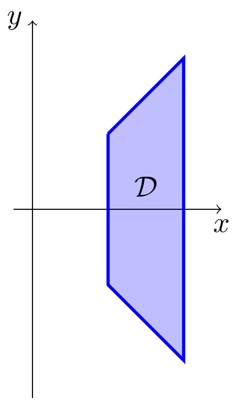
$$\int_{\mathcal{C}} \frac{z}{x^2 + y^2 + z^2}\, ds = \int_{-2\pi}^{2\pi} \frac{t}{1 + t^2} \sqrt{2}\, dt = \left[\tfrac{1}{\sqrt{2}} \ln\big(1 + t^2\big) \right]_{t=-2\pi}^{2\pi} = 0.$$

5. (10 points) Let $\mathcal{D}$ be the quadrilateral with corners $(1,1)$, $(2,2)$, $(2,-2)$, $(1,-1)$. Find

$$\iint_{\mathcal{D}} \left(x^2+y^2\right)^{-2}\, dA.$$

Hint: You may wish to use the double angle formula $\cos^2(\theta) = \frac{1}{2}\left[\cos(2\theta)+1\right]$.

Solution: Let us first sketch the region $\mathcal{D}$:



This looks familiar — we saw a similar problem on the first midterm. Converting to polar coordinates, we have

$$\begin{aligned}
\iint_{\mathcal{D}} \left(x^2+y^2\right)^{-2}\, dA &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \frac{1}{r^3}\, dr\, d\theta \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tfrac{3}{8}\cos^2\theta\, d\theta \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tfrac{3}{16}\cos(2\theta) + \tfrac{3}{16}\, d\theta \\
&= \left[\tfrac{3}{32}\sin(2\theta) + \tfrac{3}{16}\theta\right]_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \\
&= \tfrac{3}{16} + \tfrac{3}{32}\pi.
\end{aligned}$$

6. (10 points) Use the Divergence Theorem to find the flux of

$$\mathbf{F}(x,y,z) = \left\langle e^{\cos(z)}, yz + \sin(x), e^{-xy} \right\rangle$$

across the boundary of the region $\mathcal{W}$ (oriented with the outward pointing normal) where $-\sqrt{4-x^2-y^2} \leq z \leq -\sqrt{3x^2+3y^2}$.

Solution: Using spherical coordinates, our region can be written as

$$\mathcal{W} = \left\{ 0 \leq \rho \leq 2,\ 0 \leq \theta < 2\pi,\ \tfrac{5\pi}{6} \leq \phi \leq \pi \right\}.$$

Also, we compute that

$$\operatorname{div} \mathbf{F} = z.$$

Applying the Divergence Theorem, we obtain

$$\begin{aligned} \iint_{\partial \mathcal{W}} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV \\ &= \iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV \\ &= \int_0^{2\pi} \int_{\frac{5\pi}{6}}^{\pi} \int_0^2 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta \\ &= -\pi \end{aligned}$$

7. (6 points) Let $\mathcal{D} = \{x^2 + y^2 \leq 1\}$ be the unit disc and $f(t,x,y)$ be smooth function satisyfing

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}(xf) + \frac{\partial}{\partial y}(yf) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad (x,y) \in \mathcal{D}.$$

Show that

$$\frac{d}{dt} \iint_{\mathcal{D}} f(t,x,y)\,dx\,dy = \oint_{\partial\mathcal{D}} f(t,x,y)\,ds.$$

To receive full credit, you should clearly explain each step of your argument.

Solution: We compute that

$$\frac{d}{dt} \iint_{\mathcal{D}} f\,dx\,dy = \iint_{\mathcal{D}} \frac{\partial f}{\partial t}\,dx\,dy = \iint_{\mathcal{D}} \operatorname{div}\big(f\langle x,y\rangle\big)\,dx\,dy.$$

Applying the flux form of Green's Theorem, we have

$$\iint_{\mathcal{D}} \operatorname{div}\big(f\langle x,y\rangle\big)\,dx\,dy = \oint_{\partial\mathcal{D}} f\langle x,y\rangle \cdot \mathbf{n}\,ds = \oint_{\partial\mathcal{D}} f\,ds,$$

where we have used that $\langle x,y\rangle$ is the unit normal to $\partial\mathcal{D}$ at the point (x,y).

8. (16 points) Let $\mathcal{S}$ be the part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$, oriented with the upwards pointing unit normal.

   (a) Compute the surface integral
   $$\iint_{\mathcal{S}} \langle x, y, 2z \rangle \cdot d\mathbf{S}.$$

   (b) Show that
   $$\operatorname{curl}\left\langle \tfrac{1}{2}e^{z^2}, -\tfrac{1}{2}e^{z^2}, e^{xy} \right\rangle = \left\langle xe^{xy} + ze^{z^2}, -ye^{xy} + ze^{z^2}, 0 \right\rangle.$$

   (c) Using your answers to parts (a), (b), and Stokes' Theorem, find
   $$\iint_{\mathcal{S}} \left\langle x\big(1 + e^{xy}\big) + ze^{z^2}, y\big(1 - e^{xy}\big) + ze^{z^2}, 2z \right\rangle \cdot d\mathbf{S}.$$

Solution:

(a) We parameterize the surface using
$$G(r, \theta) = (r\cos\theta, r\sin\theta, r) \quad \text{for} \quad 1 \le r \le 4 \quad \text{and} \quad 0 \le \theta < 2\partial,$$
so that the tangent vectors and normal vector are
$$\mathbf{T}_r = \langle \cos\theta, \sin\theta, 1 \rangle, \quad \mathbf{T}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle, \quad \mathbf{N} = \langle -r\cos\theta, -r\sin\theta, r \rangle.$$
We then have
$$\begin{aligned}\iint_{\mathcal{S}} \langle x, y, 2z \rangle \cdot d\mathbf{S} &= \int_0^{2\pi} \int_1^4 \langle r\cos\theta, r\sin\theta, 2r \rangle \cdot \langle -r\cos\theta, -r\sin\theta, r \rangle \, dr \, d\theta \\ &= \int_0^{2\pi} \int_1^4 r^2 \, dr \, d\theta \\ &= 42\pi.\end{aligned}$$

(b) We compute that
$$\operatorname{curl}\left\langle \tfrac{1}{2}e^{z^2}, -\tfrac{1}{2}e^{z^2}, e^{xy} \right\rangle = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}e^{z^2} & -\frac{1}{2}e^{z^2} & e^{xy} \end{bmatrix} = \left\langle xe^{xy} + ze^{z^2}, -ye^{xy} + ze^{z^2}, 0 \right\rangle.$$

(c) Applying parts (a), (b), and Stokes' Theorem, we have
$$\begin{aligned}&\iint_{\mathcal{S}} \left\langle x\big(1 + e^{xy}\big) + ze^{z^2}, y\big(1 - e^{xy}\big) + ze^{z^2}, 2z \right\rangle \cdot d\mathbf{S} \\ &= \iint_{\mathcal{S}} \langle x, y, 2z \rangle \cdot d\mathbf{S} + \iint_{\mathcal{S}} \operatorname{curl}\left\langle \tfrac{1}{2}e^{z^2}, -\tfrac{1}{2}e^{z^2}, e^{xy} \right\rangle \cdot d\mathbf{S} \\ &= 42\pi + \oint_{\partial\mathcal{S}} \left\langle \tfrac{1}{2}e^{z^2}, -\tfrac{1}{2}e^{z^2}, e^{xy} \right\rangle \cdot d\mathbf{r}.\end{aligned}$$

The boundary of $\mathcal{S}$ has two components: The upper curve $\mathcal{C}_1$ with parameterization

$$\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 4\rangle \quad \text{for} \quad 0 \le t < 2\pi,$$

and the lower curve $\mathcal{C}_2$ with parameterization

$$\mathbf{r}(t) = \langle \sin t, \cos t, 1\rangle \quad \text{for} \quad 0 \le t < 2\pi.$$

Using these, we compute that

$$\oint_{\mathcal{C}_1} \left\langle \tfrac{1}{2}e^{z^2}, -\tfrac{1}{2}e^{z^2}, e^{xy}\right\rangle \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle \tfrac{1}{2}e^{16}, -\tfrac{1}{2}e^{16}, e^{16\cos t\sin t}\right\rangle \cdot \langle -4\sin t, 4\cos t, 0\rangle\, dt = 0,$$

$$\oint_{\mathcal{C}_2} \left\langle \tfrac{1}{2}e^{z^2}, -\tfrac{1}{2}e^{z^2}, e^{xy}\right\rangle \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle \tfrac{1}{2}e, -\tfrac{1}{2}e, e^{\cos t\sin t}\right\rangle \cdot \langle \cos t, \sin t, 0\rangle\, dt = 0.$$

As a consequence,

$$\iint_{\mathcal{S}} \left\langle x\big(1+e^{xy}\big) + ze^{z^2}, y\big(1-e^{xy}\big) + ze^{z^2}, 2z\right\rangle \cdot\, d\mathbf{S} = 42\pi.$$