Math 32B - Lecture 1
Fall 2018
Midterm 2
11/14/2018

Name: ____________________
TA Section: ____________________

Time Limit: 50 Minutes

Version X

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 50 points available.

Check to see if any pages are missing. Enter your name and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

Mechanics formulas

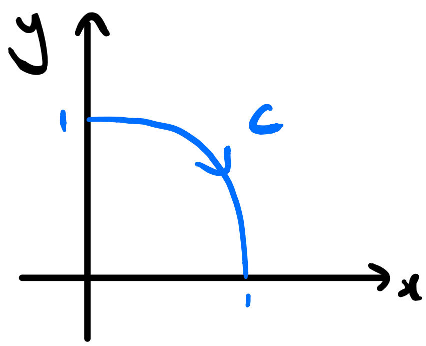
- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$
- If $\mathcal{W}$ is a solid with mass density $\delta(x,y,z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x,y,z)\,dV$
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\,\delta(x,y,z)\,dV$
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\,\delta(x,y,z)\,dV$
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\,\delta(x,y,z)\,dV$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2+z^2)\,\delta(x,y,z)\,dV$
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2+z^2)\,\delta(x,y,z)\,dV$
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2+y^2)\,\delta(x,y,z)\,dV$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$
  - The marginal probability density function of X is $p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$
  - The marginal probability density function of Y is $p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx$

1. (8 points) Find $\int_{\mathcal{C}} y\,dx - x\,dy$ where $\mathcal{C}$ is the part of the unit circle in the first quadrant, oriented **clockwise**.

FIRST, WE SKETCH C:



WE PARAMETERIZE C USING

$$\vec{r}(t) = \langle \sin t, \cos t \rangle \quad \text{FOR} \quad 0 \le t \le \tfrac{\pi}{2}$$

WE THEN COMPUTE

$$\vec{r}\,'(t) = \langle \cos t, -\sin t \rangle$$

THEN, WE HAVE

$$\int_C y\,dx - x\,dy = \int_0^{\frac{\pi}{2}} \langle \cos t, -\sin t \rangle \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \langle \cos t, -\sin t \rangle \cdot \langle \cos t, -\sin t \rangle\,dt$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t + \sin^2 t\,dt$$

$$= \int_0^{\frac{\pi}{2}} dt$$

$$= \pi/2 \,.$$

2. (10 points) Let $\mathbf{F}(x,y,z) = \langle \cos(yz), -xz\sin(yz)+1, -xy\sin(yz)+2z \rangle$ and $\mathcal{C}$ be a smooth curve from $(0,0,1)$ to $(1, \frac{\pi}{2}, 1)$. Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

WE SEEK A POTENTIAL FUNCTION $f(x,y,z)$ FOR $\vec{F}(x,y,z)$, SO THAT $\vec{F} = \nabla f$.

COMPARING THE x-COMPONENTS,

$$\frac{\partial f}{\partial x} = \cos(yz)$$

SO $f(x,y,z) = x\cos(yz) + g(y,z)$

FOR SOME FUNCTION $g(y,z)$.

COMPARING THE y-COMPONENTS,

$$\frac{\partial f}{\partial y} = -xz\sin(yz)+1$$

$$\cancel{-xz\sin(yz)} + \frac{\partial g}{\partial y} = \cancel{-xz\sin(yz)}+1$$

$$\frac{\partial g}{\partial y} = 1$$

SO $g(y,z) = y + h(z)$

FOR SOME FUNCTION $h(z)$.

FINALLY, COMPARING THE z-COMPONENTS,

$$\frac{\partial f}{\partial z} = -xy\sin(yz) + 2z$$

$$\cancel{-xy\sin(yz)} + h'(z) = \cancel{-xy\sin(yz)} + 2z$$

$$h'(z) = 2z$$

SO $h(z) = z^2 +$ CONSTANT.

SO $f(x,y,z) = x\cos(yz) + y + z^2$

IS A POTENTIAL FUNCTION FOR $\vec{F}$.

APPLYING THE FUNDAMENTAL THEOREM FOR CONSERVATIVE VECTOR FIELDS:

$$\int_C \vec{F}\cdot d\vec{r} = f\left(1,\tfrac{\pi}{2},1\right) - f(0,0,1)$$

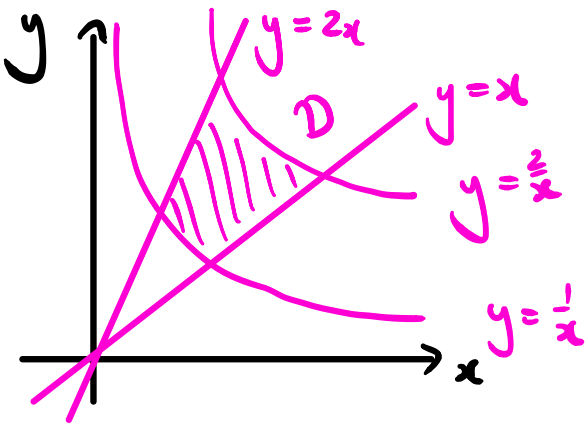
$$= \frac{\pi}{2} + 1 - 1$$

$$= \pi/2.$$

3. (16 points) Let $\mathcal{D}$ be a lamina contained in the region $\left\{1 \le xy \le 2,\ 1 \le \frac{y}{x} \le 2,\ x \ge 0,\ y \ge 0\right\}$ (where distance is measure in metres) with density $\delta(x,y) = \frac{2y}{x}$ kilograms per square metre. Find the moment of inertia of $\mathcal{D}$ about the x-axis.

$\Big($*Hint: You might wish to use the variables* $u = xy$ *and* $v = \frac{y}{x}$.$\Big)$

FIRST SKETCH D:



IF WE TAKE $u = xy$, $v = y/x$, THEN

$$D = \{1 \le u \le 2,\ 1 \le v \le 2\}$$

THE MOMENT OF INERTIA ABOUT THE x-AXIS IS

$$I_x = \iint_D y^2\, \delta(x,y)\, dA.$$

COMPUTE THE JACOBIAN:

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} y & x \\ -y/x^2 & \frac{1}{x} \end{bmatrix} = 2\, y/x = 2v$$

THEN,

$$I_x = \int_1^2 \int_1^2 uv \cdot 2v \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du\,dv$$

$$= \int_1^2 \int_1^2 uv \cdot \cancel{2v} \cdot \frac{1}{\cancel{2v}} \, du\,dv$$

$$= \int_1^2 \int_1^2 uv \, du\,dv$$

$$= \int_1^2 \left[\tfrac{1}{2} u^2 v \right]_1^2 dv$$

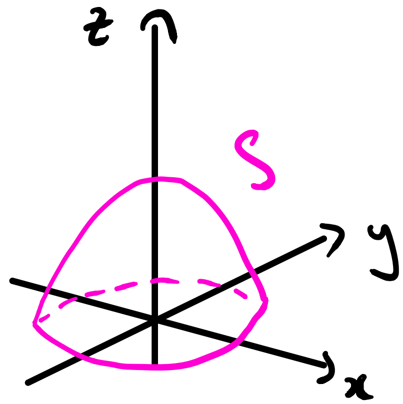
$$= \int_1^2 \tfrac{3}{2} v \, dv$$

$$= \left[\tfrac{3}{4} v^2 \right]_1^2$$

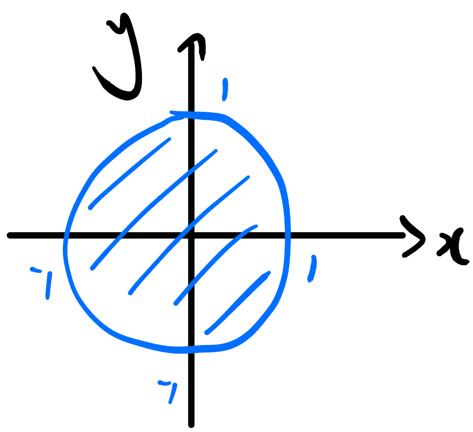
$$= 9/4 \quad \text{kg m}^2 .$$

4. (16 points) Let the surface $\mathcal{S}$ be the part of the paraboloid $z = 1 - x^2 - y^2$ where $z \geq 0$ and let $f(x, y, z) = \sqrt{(1-z) + 4(1-z)^2}$. Find $\iint_{\mathcal{S}} f(x, y, z)\, dS$.

FIRST SKETCH S:



ALSO SKETCH PROJECTION ONTO (x,y)-PLANE:



PARAMETERIZE S USING

$$G(r, \theta) = \left(r\cos\theta,\ r\sin\theta,\ 1 - r^2\right)$$

$0 \leq r \leq 1$ AND $0 \leq \theta < 2\pi$.

NEXT, COMPUTE

$$\vec{T}_r = \frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$$

$$\vec{T}_\theta = \frac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\vec{N} = \vec{T}_r \times \vec{T}_\theta = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

SO $\|\vec{N}\| = \sqrt{4r^4 + r^2}$

THEN

$$\iint_S f \, dS = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 4r^4} \cdot \|\vec{N}\| \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^2 + 4r^4 \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} + \frac{4}{5} \, d\theta$$

$$= \frac{34}{15}\pi$$