

Math 32A  
Fall 2016  
Midterm 1  
November 7th, 2016

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Section: (circle one) 3A (Tue) 3B (Thu) w/Ioannis  
3C (Tue) 3D (Thu) w/John  
3E (Tue) 3F (Thu) w/Tianqi

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you make use of a theorem from lecture (or the textbook) in the course of your work, make sure to indicate which theorem was used and how it was used. Failure to do so will result in deduction of points.
- Write your solutions in the space below the questions. If you need more space, use the back of the page and clearly indicate when you have done this. Do not turn in your scratch paper.

Do not write in the table to the right.

mean: 61.1

median: 65

high: 79

A: 70-80

B: 62-69

C: 40-61

D/F: 0-39

| Problem | Points | Score |
|---------|--------|-------|
| 1       | 20     | 20    |
| 2       | 20     | 20    |
| 3       | 20     | 13    |
| 4       | 20     | 18    |
| Total:  | 80     | 71    |

1. (20 points) Consider the curve  $C$  parameterized by

$$\mathbf{r}(t) = \langle 3t, \cos(4t), \sin(4t) \rangle$$

for  $t \geq 0$ .

- (a) (5 points) Calculate the unit tangent vector  $\mathbf{T}$  and unit normal vector  $\mathbf{N}$  to the curve  $C$  at  $\mathbf{r}(t)$  for  $t \geq 0$ .

$$\mathbf{r}(t) = \langle 3t, \cos(4t), \sin(4t) \rangle$$

$$\mathbf{r}'(t) = \langle 3, -4\sin(4t), 4\cos(4t) \rangle$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{9 + 16(\sin^2(4t) + \cos^2(4t))} \\ &= \sqrt{25} = 5 \end{aligned}$$

$$\therefore \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\begin{aligned} &= \frac{1}{5} \langle 3, -4\sin(4t), 4\cos(4t) \rangle \\ &= \left\langle \frac{3}{5}, -\frac{4}{5}\sin(4t), \frac{4}{5}\cos(4t) \right\rangle \end{aligned}$$

$$\begin{aligned} \mathbf{T}'(t) &= \left\langle 0, -\frac{16}{5}\cos(4t), \frac{16}{5}\sin(4t) \right\rangle \\ \|\mathbf{T}'(t)\| &= \sqrt{\left(\frac{16}{5}\right)^2 (\cos^2(4t) + \sin^2(4t))} \\ &= \frac{16}{5} \end{aligned}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

$$\begin{aligned} &= \frac{5}{16} \langle 0, -16\cos(4t), 16\sin(4t) \rangle \\ &= \langle 0, -\cos(4t), \sin(4t) \rangle \end{aligned}$$

- (b) (5 points) Calculate the arc length  $s(t)$  of the parameterization  $\mathbf{r}(t)$  as a function of  $t$  for  $t \geq 0$ .

$$\begin{aligned} s(t) &= \int_0^t \|\mathbf{r}'(t)\| dt \\ &= \int_0^t 5 dt = 5t \text{ units} \end{aligned}$$

- (c) (5 points) Find the arc length parameterization  $\mathbf{r}_1(s)$  of the curve  $C$ .

$$\begin{aligned} \text{from (b), } s &= 5t = g(t) \\ t &= \frac{1}{5}s = g^{-1}(s) \end{aligned}$$

$$\begin{aligned} \mathbf{r}_1(s) &= \mathbf{r}(g^{-1}(s)) = \mathbf{r}\left(\frac{s}{5}\right) = \left\langle 3\left(\frac{s}{5}\right), \cos\left(\frac{4s}{5}\right), \sin\left(\frac{4s}{5}\right) \right\rangle \\ &= \left\langle \frac{3s}{5}, \cos\left(\frac{4s}{5}\right), \sin\left(\frac{4s}{5}\right) \right\rangle \end{aligned}$$

- (d) (5 points) Calculate the curvature  $\kappa(s)$  of the curve  $C$  at a point  $\mathbf{r}_1(s)$ .

Answer at back of this page.

Q 1 (d)

$$H(s) = \left\| \frac{dT}{ds} \right\| = \left\| \frac{dT}{dt} \times \frac{dt}{ds} \right\|$$

$$= \|T'(t) \times \frac{1}{5}\|$$

$$= \left\| \left\langle 0, -\frac{16}{5} \cos(4t), -\frac{16}{5} \sin(4t) \right\rangle \times \frac{1}{5} \right\|$$

$$= \sqrt{\left(\frac{16}{25}\right)^2 (\cos^2(4t) + \sin^2(4t))}$$

$$= \frac{16}{25} \neq$$

2. (20 points) Consider the quadratic surface given by the equation

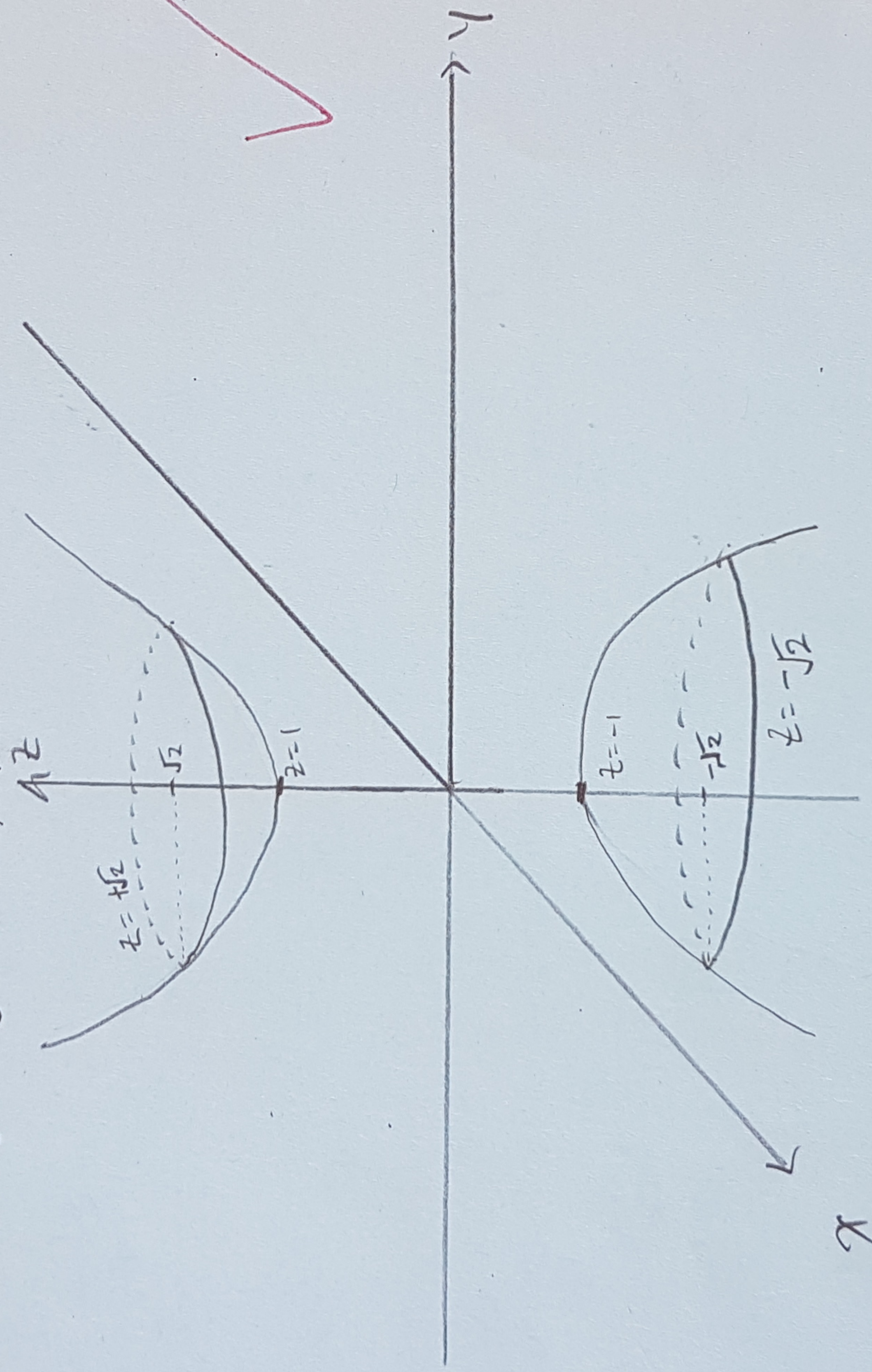
$$-x^2 - y^2 + 4z^2 = 4. \quad (1)$$

- (a) (2 points) Classify the surface. That is, say what type of quadratic surface is given by

Equation (1).  $4z^2 = 4 + x^2 + y^2 \implies \left(\frac{z}{1}\right)^2 - 1 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2$

$z^2 - 1 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 \implies \therefore \text{Hyperboloid (2-sheet)}$

- (b) (8 points) Sketch the surface and, included in your sketch, draw the horizontal traces corresponding to  $z = 1$ ,  $z = -1$ ,  $z = \sqrt{2}$  and  $z = -\sqrt{2}$ . Be sure to label your axes. You are free to draw additional traces (to help you in your sketch); however, you **must** label the traces corresponding to  $z = \pm 1, \pm\sqrt{2}$ .



- (c) (5 points) Give a parameterization  $\mathbf{r}(t)$  of the trace corresponding to  $z = \sqrt{2}$ .

$$-x^2 - y^2 + 4(\sqrt{2})^2 = 4$$

$$x^2 + y^2 = 4$$

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \sqrt{2} \rangle$$

- (d) (5 points) Using your parameterization, calculate the curvature  $\kappa(t)$  of the trace corresponding to  $z = \sqrt{2}$  at  $\mathbf{r}(t)$ .

$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$$

$$\mathbf{r}''(t) = \langle -2\cos t, -2\sin t, 0 \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

$$\therefore \kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\langle 0, 0, 4\sin^2 t + 4\cos^2 t \rangle\|}{2^3} = \frac{4}{8} = \frac{1}{2}$$

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3. (20 points) Define a function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

for  $(x, y) \in \mathbb{R}^2$ .

(a) (8 points) Determine the set of points at which  $f$  is continuous. Justify your answer.

$$f(x, y) = \begin{cases} \frac{x}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

For all  $(x, y)$  exclude  $(x, y) = (0, 0)$ ,  $f(x, y)$  exists continuously because no value of  $(x, y)$  is excluded naturally.

sub  $y=x$  into  $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{x}{2x^2} = \lim_{x \rightarrow 0} \frac{1}{2x} = \infty \neq 0$  (from  $f(x, y) = 0$  in given definition)

(L'Hopital's Rule)

$\therefore f$  is discontinuous at  $(x, y) = (0, 0)$

$\therefore$  Set:  $D = \{(x, y) \in \mathbb{R}^2, (x, y) \neq (0, 0)\}$

(b) (10 points) Compute the partial derivatives  $f_x$  and  $f_y$  where they exist.

$$f_x(x, y) = \frac{\partial}{\partial x} \frac{x}{x^2+y^2} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

Check

$$= \frac{x^2+y^2 - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$f_y(x, y) = \frac{\partial}{\partial y} \frac{x}{x^2+y^2} = \frac{\partial}{\partial y} x(x^2+y^2)^{-1}$$

Chain Rule

$$= x(-1)(x^2+y^2)^{-2}(2y)$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

(c) (2 points) Are the partial derivatives  $f_x$  and  $f_y$  continuous at all points in  $\mathbb{R}^2$ ?

No,  $f_x$  is not continuous at  $(0, 0)$  because it is undefined  $\frac{0-0}{(0+0)^2} = \frac{0}{0}$

No,  $f_y$  is not continuous at  $(0, 0)$  because it is undefined  $\frac{0}{(0+0)^2} = \frac{0}{0}$

4. (20 points) In what follows,  $f$  is a two-variable function with domain  $D \subseteq \mathbb{R}$ . You may assume that  $f$  is defined near  $(0, 0)$ .

(a) (10 points) TRUE OR FALSE (circle one, 2 points each)

- True  False  According to Kepler's laws, planets travel in ellipses with the sun at one focus.
- True  False  The curvature  $\kappa$  of a curve  $C$  is always non-negative.
- True  False  To verify that the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  exists and is equal to  $L$ , it suffices to show that  $f(x,y)$  tends to  $L$  as  $(x,y)$  approaches  $(0,0)$  along all lines of the form  $y = mx$ . *curved lines.*
- True  False  For  $f$  to be continuous at  $(0,0)$ , it is necessary that  $(0,0) \in D$ .
- True  False  Contour lines corresponding to distinct  $z$ -values of  $f$  can never intersect.

(b) (5 points) Determine whether the following limit exists and, if it does, compute it. Justify your answer.

$$\lim_{(x,y) \rightarrow (0,0)} 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right)$$

By squeeze theorem,  $\therefore$  the limit exists,  $L = 0$

$$-4x^2y^4 \leq 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right) \leq 4x^2y^4$$

$$\lim_{(x,y) \rightarrow (0,0)} -4x^2y^4 \leq \lim_{(x,y) \rightarrow (0,0)} 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right) \leq \lim_{(x,y) \rightarrow (0,0)} 4x^2y^4$$

$$0 \leq \lim_{(x,y) \rightarrow (0,0)} 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right) \leq 0$$

By Squeeze theorem

$$\lim_{(x,y) \rightarrow (0,0)} 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right) = 0$$

(c) (5 points) Consider the function

$$f(x,y) = x^2y + \frac{\cos\left(\frac{x^4}{x^2+1}\right)}{2 + \sin(x^2)}$$

defined for all  $(x,y) \in \mathbb{R}^2$ . Calculate  $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$  at the point  $(a,b)$  (You may assume that  $f_{xy}$  and  $f_{yx}$  exist and are continuous on  $\mathbb{R}^2$ ).

Since  $f_{xy}$  &  $f_{yx}$  exist continuously on  $\mathbb{R}^2$ ,  $\therefore$  Clairaut's Theorem applies.

ie.  $f_{xy} = f_{yx}$

$$f_y = \frac{\partial}{\partial y} x^2y + \frac{\cos\left(\frac{x^4}{x^2+1}\right)}{2 + \sin(x^2)}$$

$$= x^2$$

$$f_{yx} = \frac{\partial}{\partial x} x^2 = 2x$$

$$\therefore f_{xy} = f_{yx} = 2x$$

at point  $(a,b)$ ,  $f_{xy} = 2(a) = 2a$