Math 32A
Fall 2016
Midterm 1
October 17th, 2016
Time Limit: 50 Minutes

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Section: 3A 3B 3C 3D 3E 3F (circle one)

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Write your solutions in the space below the questions. If you need more space, use the back of the page and clearly indicate when you have done this. Do not turn in your scratch paper.

Do not write in the table to the right.

Problem	Points	Score
1	25	24
2	25	25
3	25	25
4	25	20
Total:	100	94

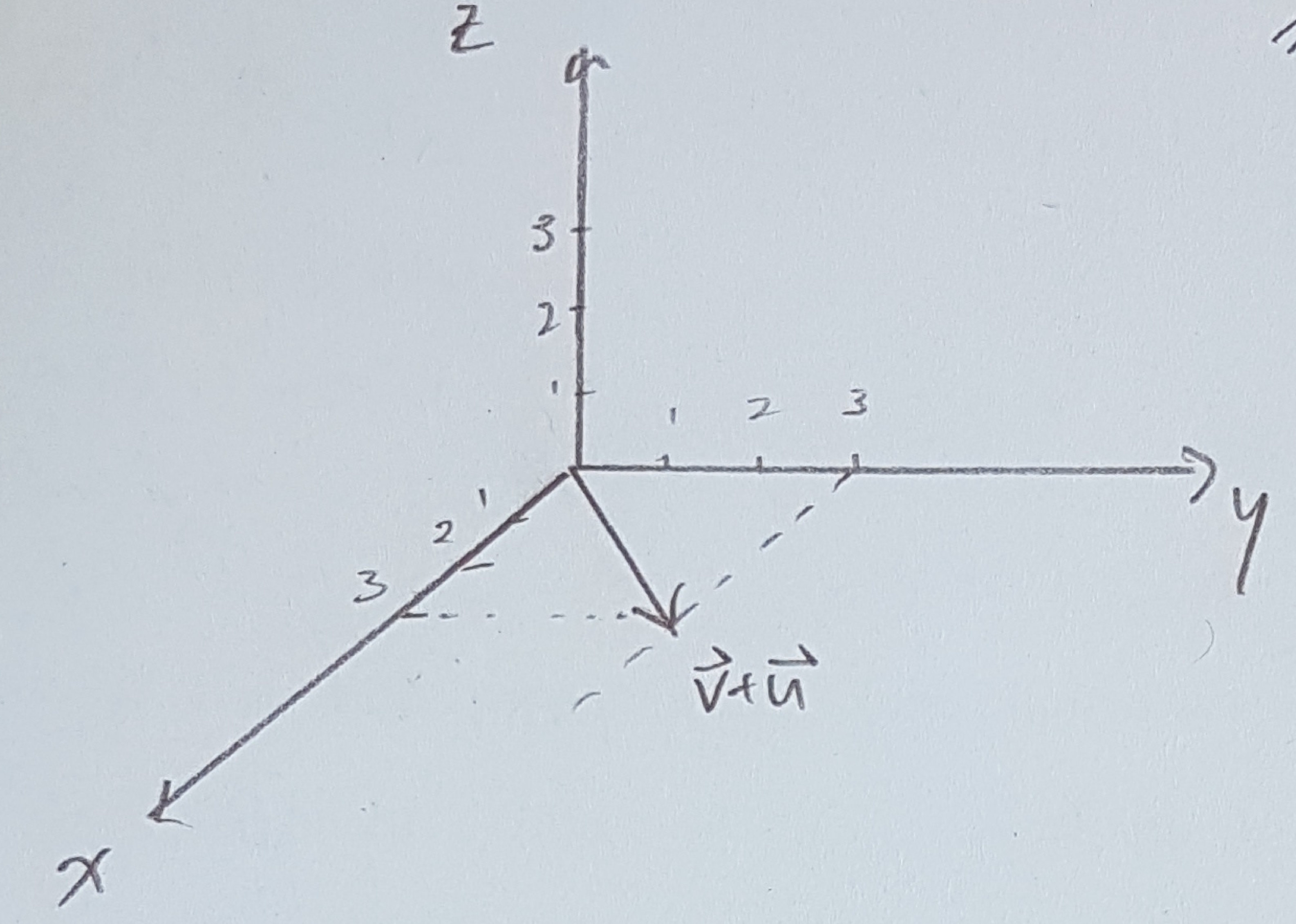
49

45

1. Let $\mathbf{v} = \langle 1, 2, 1\rangle$ and $\mathbf{u} = \langle 2, 1, -1\rangle$.

(a) (5 points) Draw the vector $\mathbf{v} + \mathbf{u}$.

$\vec{v}+\vec{u} = \langle 1+2, 2+1, 1-1\rangle = \langle 3, 3, 0\rangle$ //



(b) (10 points) Find the parallel projection of $\mathbf{u}$ onto $\mathbf{v}$, i.e., find the vector $\mathbf{u}_{||\mathbf{v}}$.

$$\vec{u}_{||v} = \frac{u \cdot v}{v \cdot v}\vec{v} = \frac{\langle 2,1,-1\rangle \cdot \langle 1,2,1\rangle}{\langle 1,2,1\rangle \cdot \langle 1,2,1\rangle}\langle 1,2,1\rangle$$

$$= \frac{2+2-1}{1+4+\cancel{2}}\langle 1,2,1\rangle$$

$$= \frac{3}{\cancel{7}\,6}\langle 1,2,1\rangle$$

$$= \left\langle \frac{3}{7}, \frac{6}{7}, \frac{3}{7}\right\rangle \ //$$

-1

(c) (10 points) Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{u}$.

27
9×3

Area of parallelogram $= \| v \times u \|$

$= \| \langle 1,2,1\rangle \times \langle 2,1,-1\rangle \|$

$= \left\| \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\hat{k} \right\|$

$= \| -3i + 3j - 3k \|$

$= \sqrt{9+9+9}$

$= 3\sqrt{3}$ units^2 //

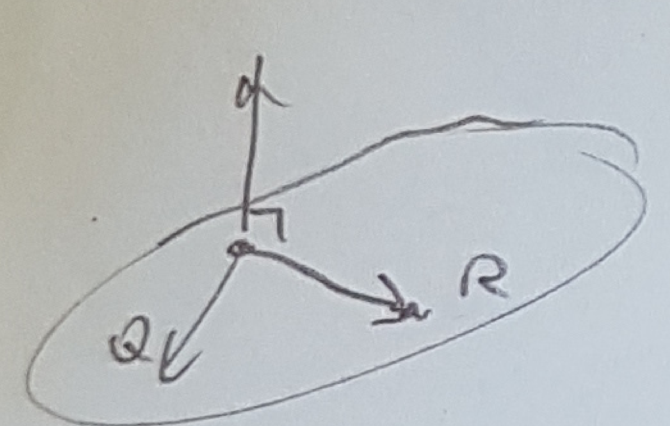
2. Given the points $P = (1,2,3)$, $Q = (3,4,4)$ and $R = (2,2,4)$, find:

(a) (5 points) The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$$\overrightarrow{PQ} = \langle 3-1, 4-2, 4-3 \rangle = \langle 2,2,1 \rangle$$
$$\overrightarrow{PR} = \langle 2-1, 2-2, 4-3 \rangle = \langle 1,0,1 \rangle$$

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\| \cos\theta$$
$$2+0+1 = \sqrt{9}\sqrt{2}\cos\theta$$
$$\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ //$$

(b) (10 points) A unit vector perpendicular to the plane containing P, Q and R



Perpendicular vector = cross product of $\overrightarrow{PQ}$ & $\overrightarrow{PR}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2,2,1 \rangle \times \langle 1,0,1 \rangle$$
$$= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix}\hat{k}$$
$$= 2\hat{i} - \hat{j} - 2\hat{k} = \langle 2,-1,-2 \rangle$$

Unit vector $= \left|\frac{1}{\sqrt{9}}\right| \langle 2,-1,-2 \rangle = \langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \rangle //$

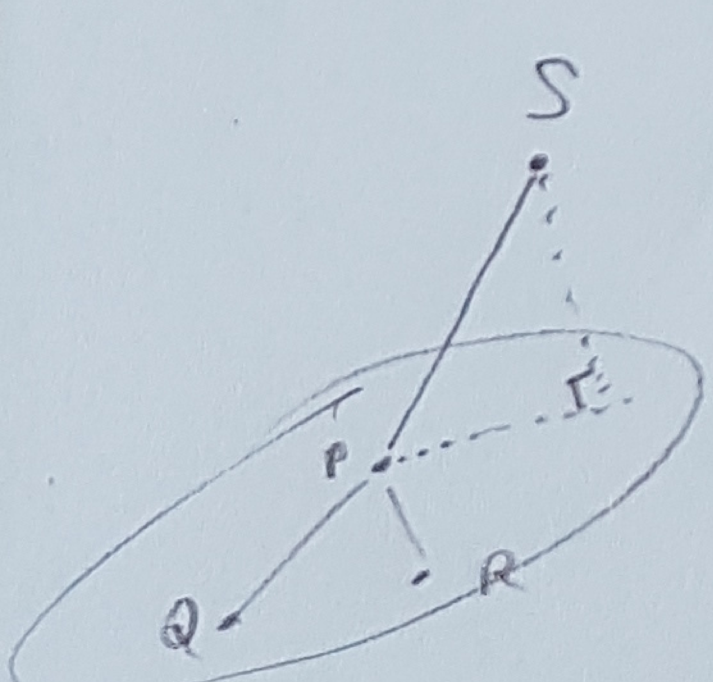
(c) (5 points) The equation of the plane containing P, Q and R

A Point on plane is $P = (1,2,3)$

$$\langle 2,-1,-2 \rangle \cdot \langle 1,2,3 \rangle = 2-2-6 = -6$$

$\therefore$ equation: $\langle 2,-1,-2 \rangle \cdot \langle x,y,z \rangle = -6$ //

(d) (5 points) The distance from the plane containing P, Q and R to the point $S = (0,1,0)$



vector ⊥ to plane: $\langle 2,-1,-2 \rangle = \hat{u}$

$\overrightarrow{PS} = \langle 0-1, 1-2, 0-3 \rangle = \langle -1,-1,-3 \rangle$

distance of P to plane $= \|\overrightarrow{PS}_{u\hat{n}}\|$

$$= \left\| \frac{\langle -1,-1,-3 \rangle \cdot \langle 2,-1,-2 \rangle}{4+1+4} \langle 2,-1,-2 \rangle \right\|$$
$$= \left\| \frac{-2+1+6}{9} \langle 2,-1,-2 \rangle \right\|$$
$$= \left\| \frac{10}{9}, -\frac{5}{9}, -\frac{10}{9} \right\| = \sqrt{\frac{100}{81} + \frac{25}{81} + \frac{100}{81}} = \sqrt{\frac{225}{81}} = \frac{5}{3} \text{ units} //$$

$15 \times 15 = 225$

$\frac{225}{5} = 45$

45×5

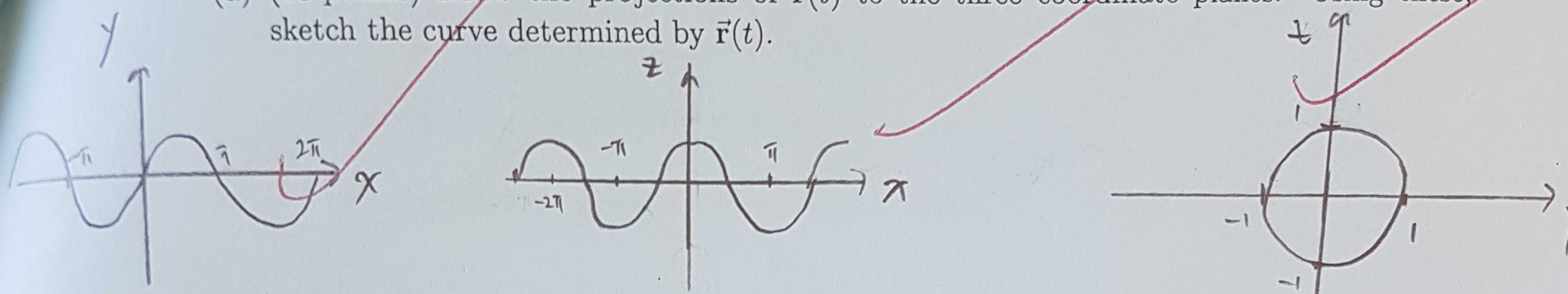
$9 \times 5 \times 5$

$25 \times 20 = 500$

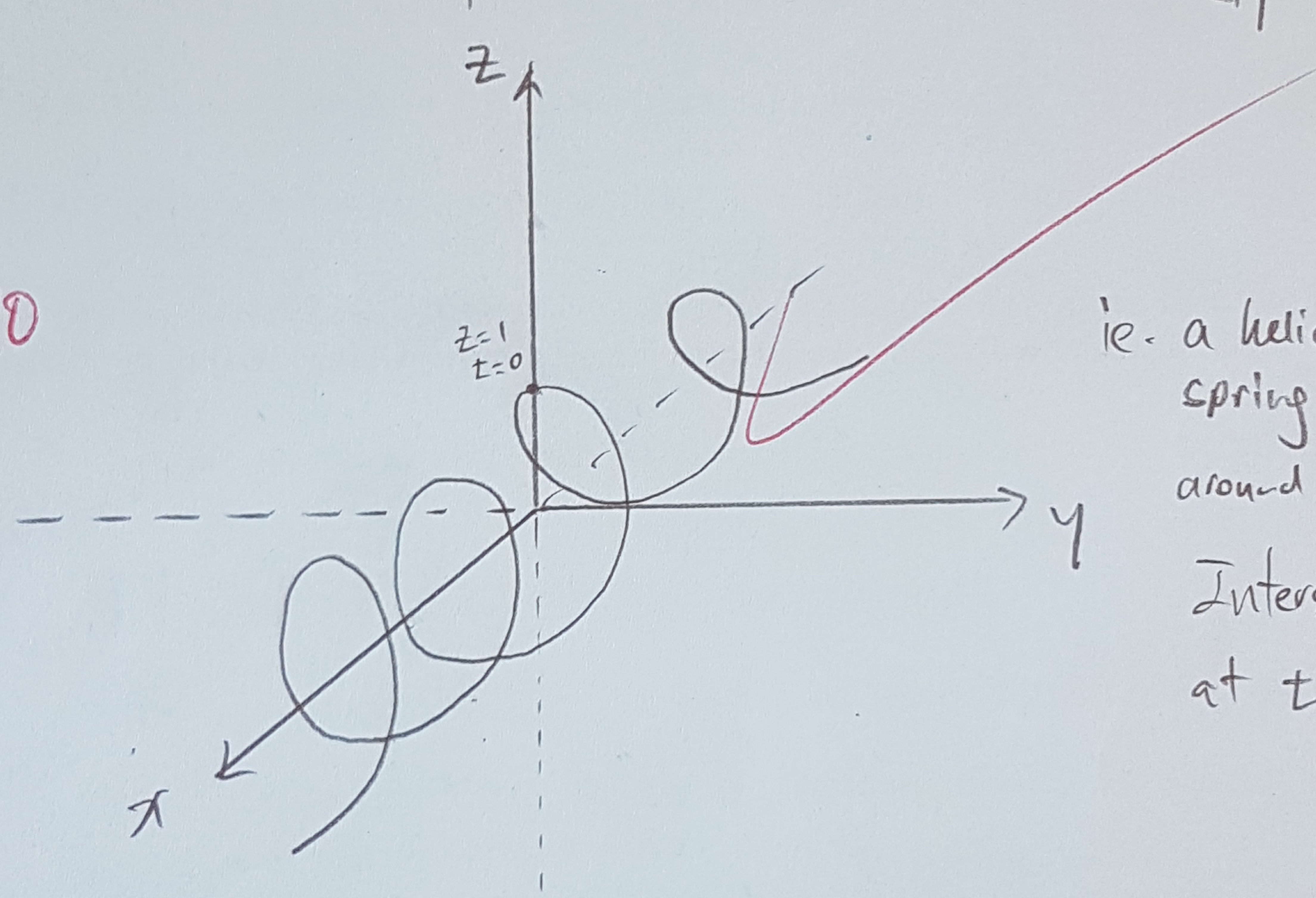
25

3. Consider the vector-valued function $\vec{\mathbf{r}}(t) = \langle t, \sin t, \cos t\rangle$ for $-\infty < t < \infty$.

(a) (10 points) Draw the projections of $\vec{\mathbf{r}}(t)$ to the three coordinate planes. Using these, sketch the curve determined by $\vec{\mathbf{r}}(t)$.



10/10



ie. a helical spring extending around x-axis.

Intersects z-axis at t=0, z=1

(b) (15 points) Find a vector parameterization of the tangent line to $\vec{\mathbf{r}}(t)$ at $t = \pi/6$.

$\frac{180}{6} = 30°$

sin 30 45 60

$\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$

15/15

tangent line's gradient vector $= \vec{r}'(t)$

$= \langle 1, \cos t, -\sin t\rangle$

at $t = \pi/6$, gradient vector $= \langle 1, \cos\frac{\pi}{6}, -\sin\frac{\pi}{6}\rangle$

$= \langle 1, \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle$

$\therefore$ tangent line $=$ (scalar x vector gradient) + (translation to point (non-origin))

$= k\langle 1, \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle + \vec{r}(\frac{\pi}{6})$

$= k\langle 1, \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle + \langle \frac{\pi}{6}, \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = \langle \frac{\pi}{6}+k, \frac{1}{2}+\frac{\sqrt{3}}{2}k, \frac{\sqrt{3}}{2}-\frac{1}{2}k\rangle$

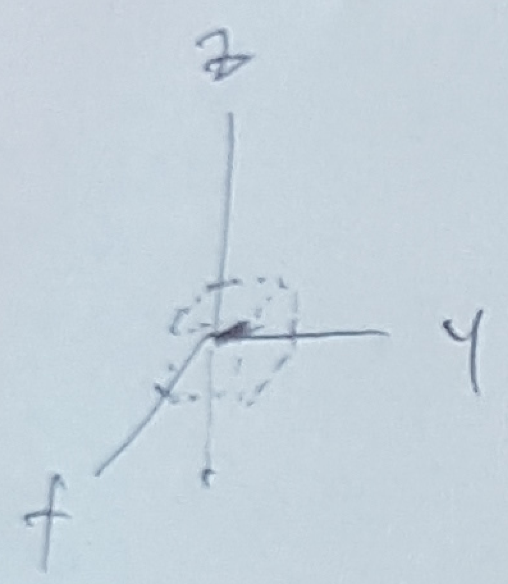
$\therefore$ line. $\begin{cases} x = \frac{\pi}{6} + k \\ y = \frac{1}{2} + \frac{\sqrt{3}}{2}k \\ z = \frac{\sqrt{3}}{2} - \frac{1}{2}k \end{cases}$ //

where k is a scalar constant.

20/25

4. In answering the following question, recall that the zero vector is, by convention, orthogonal to every vector.

(a) (15 points) TRUE OR FALSE (circle one)



The dot product between two vectors is a vector.	TRUE	FALSE
The cross product between two vectors is a vector.	TRUE	FALSE
Two vectors $\mathbf{v}$ and $\mathbf{u}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{u} = 0$.	TRUE	FALSE
For any three vectors $\mathbf{v}, \mathbf{u}$ and $\mathbf{w}$, $\mathbf{v} \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \times \mathbf{w}$.	TRUE	FALSE
There exists a vector $\mathbf{v}$ such that $\mathbf{v} \times \langle 1,1,1\rangle = \langle 1,2,0\rangle$.	TRUE	FALSE

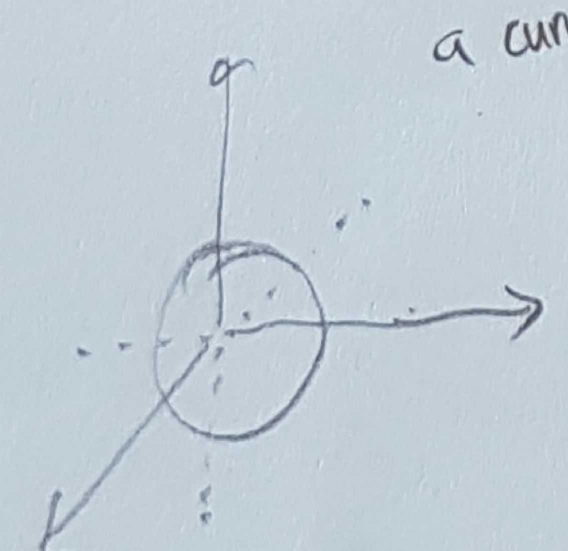
$1+2+0=3$

(b) (10 points) Let $\vec{\mathbf{r}}(t)$ be differentiable and let C be a constant. Show that the following statement is true. Your reasoning/justification should be well-written and clear.

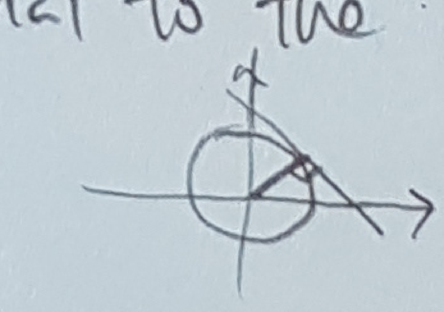
If $\|\vec{\mathbf{r}}(t)\| = C$ for all t, then $\frac{d}{dt}\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}'(t)$ is orthogonal to $\vec{\mathbf{r}}(t)$ for all t.

If $\|\vec{r}(t)\| = C$ for all t, that means $\vec{r}(t)$ must trace a path on the surface of a sphere centered around the point of origin, because a sphere by definition is a curved plane of constant distance from origin, C.

5/10



For any point on the sphere, the tangent must be orthogonal to the line from center to surface of sphere.



$\frac{d}{dt}\vec{r}(t) = \vec{r}'(t)$ gives a direction vector for the tangent to $\vec{r}(t)$, which is a path on the surface of the sphere. Thus it follows that $\vec{r}'(t)$ is definitely orthogonal to $\vec{r}(t)$ for all t.

why does the $\vec{r}'$ coincide with a vector tangent to the surface

why.

You need to show more. see online solution