MATH 32A Fall 2017
Midterm 2

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This exam contains 7 pages (including this page) and 5 questions. Total of points is 100. Work neatly and show all your work, including all intermediate steps and also explain in words how you are solving a problem. Partial credits are available for most problems. Calculators are not allowed in this exam. You have 50 minutes.

Grade Table (for instructor use only)

Question	Points	Score
1	20	20
2	20	16
3	20	6
4	20	18
5	20	16
Total:	100	76

$-2x e^{-x^2 y^{-1}}$

ce

$-2x e^{-x^2 y^{-1}} \left(\frac{x^2}{y^2}\right)$

$-\frac{2x}{y^2} e^{-\frac{x^2}{y}}$

$-\frac{x^2}{y}$ $\quad -x^2 y^{-1}$ $\quad \frac{x^2}{y^2}$

1. (20 points) (a) (10 points) Compute the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

(b) (10 points) Find $f(x,y) = e^{\frac{-x^2}{y}}$. Compute $\frac{\partial^2 f}{\partial x \partial y}(1,1)$ and $\frac{\partial^2 f}{\partial y \partial x}(1,1)$

$-\frac{2xe^{-\frac{x^2}{y}}}{y}$

a) $\frac{\delta f}{\delta x} = \frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2}$

$= \frac{2x^3 + 2xy^2 - 2x^3 + 2xy^2}{(x^2+y^2)^2}$

$\boxed{\frac{\delta f}{\delta x} = \frac{4xy^2}{(x^2+y^2)^2}}$ ✓

$\frac{\delta f}{\delta y} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2}$

$= \frac{-2yx^2 - 2y^3 - 2x^2y + 2y^3}{(x^2+y^2)^2}$

$\boxed{\frac{\delta f}{\delta y} = \frac{-4yx^2}{(x^2+y^2)^2}}$ ✓

b) $\frac{\delta f}{\delta x} = -\frac{2}{y} x e^{-\frac{x^2}{y}} = \frac{-2xe^{-\frac{x^2}{y}}}{y}$

$\frac{\delta^2 f}{\delta x \delta y} = \frac{y\left(-\frac{2x^3}{y^2} e^{-\frac{x^2}{y}}\right) + 2xe^{-\frac{x^2}{y}}}{y^2}$

$\frac{\delta^2 f}{\delta x \delta y}(1,1) = \frac{(1)((-2)e^{-1}) + (2)e^{-1}}{1}$

$= \frac{-\frac{2}{e} + \frac{2}{e}}{1}$ ✓

$\boxed{\frac{\delta^2 f}{\delta x \delta y}(1,1) = 0}$

$\boxed{\frac{\delta^2 f}{\delta y \delta x}(1,1) = 0}$ ✓

2. (20 points) (a) (5 points) Find the arc-length parameterization of the helix

$$\vec{r}(t) = (\cos t, \sin t, \sqrt{2}t),$$

starting at $t = 0$.

(b) (5 points) Find the curvature of $\vec{r}(t)$ at $t = \frac{\pi}{4}$ by definition.

(c) (10 points) Find the curvature of $\vec{r}(t)$ at $t = \frac{\pi}{4}$ by the formula $\kappa(t) = \frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3}$.

a) $r'(t) = \langle -\sin t, \cos t, \sqrt{2} \rangle$

$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + (\sqrt{2})^2}$

$\|r'(t)\| = \sqrt{2+1}$

$s = \int_0^t \sqrt{3}\, dt$

$s = t\sqrt{3}$

$\frac{s}{\sqrt{3}} = g^{-1}(s)$

$\sqrt{2}$

$r_1(s) = \langle \cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \sqrt{2}\cdot\frac{s}{\sqrt{3}} \rangle$

$\boxed{r_1(s) = \langle \cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \frac{\sqrt{2}s}{\sqrt{3}} \rangle}$

b) $T(t) = \frac{\langle -\sin t, \cos t, \sqrt{2} \rangle}{\sqrt{3}}$

$T(t) = \left\langle \frac{-\sin t}{\sqrt{3}}, \frac{\cos t}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \right\rangle$

$\kappa(t) = \left\| \frac{dT}{dt} \right\|$

$\frac{dT}{dt} = \left\langle \frac{-\cos t}{\sqrt{3}}, \frac{-\sin t}{\sqrt{3}}, 0 \right\rangle$

$\kappa(t) = \sqrt{\left(\frac{-\cos t}{\sqrt{3}}\right)^2 + \left(\frac{-\sin t}{\sqrt{3}}\right)^2}$

$= \sqrt{\frac{\cos^2 t + \sin^2 t}{3}}$

$= \sqrt{\frac{1}{3}}$

c) $r'(t) = \langle -\sin t, \cos t, \sqrt{2} \rangle$

$r''(t) = \langle -\cos t, -\sin t, 0 \rangle$

$r'(t) \times r''(t) = \langle \sqrt{2}\sin t, -\sqrt{2}\cos t, \sin^2 t + \cos^2 t \rangle$

$= \langle \sqrt{2}\sin t, -\sqrt{2}\cos t, 1 \rangle$

$\kappa(t) = \frac{\sqrt{(\sqrt{2}\sin t)^2 + (\sqrt{2}\cos t)^2 + (1)^2}}{\left(\sqrt{(-\sin t)^2 + (\cos t)^2 + (\sqrt{2})^2}\right)^3}$

$= \frac{\sqrt{2(\sin^2 t + \cos^2 t) + 1}}{\left(\sqrt{\sin^2 t + \cos^2 t + 2}\right)^3}$

$\sqrt{2}^3$ $\quad = \frac{\sqrt{3}}{(\sqrt{2})^3}$

$2^{3/2}$

3. (20 points) (a) (10 points) Use $\epsilon - \delta$ to prove $\lim_{(x,y)\to(1,2)} x^2 + y^2 = 5$.

(b) (5 points) Calculate the limit if it exists or indicate that it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + xy^2}{x^2+y^2}.$$

(c) (5 points) Calculate the limit if it exists or indicate that it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}.$$

a) $\lim_{(x,y)\to(1,2)} x^2+y^2 = 5$

For If $\epsilon>0$, there exists $\delta>0$ s.t.

$0<\sqrt{(x-1)^2+(y-2)^2}<\delta$ then $|x^2+y^2-5|<\epsilon$

$|x^2-(1)^2+y^2-(2)^2|<\epsilon$

$\sqrt{(x-1)^2}+\sqrt{(y-2)^2} \geq \sqrt{(x-1)^2+(y-2)^2}$

$|x^2-(1)|+|y^2-(2)| \geq \sqrt{(x-1)^2+(y-2)^2}$

$\sqrt{(x-1)^2+(y-2)^2} \leq |x^2-1|+|y^2-2| \leq \epsilon$

2

b) $x = r\cos\theta$

$y = r\sin\theta$

$\cos^2\theta$

$1-\sin^2\theta$ +

$\sin\theta$

$\lim_{r\to0} \frac{r^2\cos^2\theta\, r\sin\theta + r\cos\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$\lim_{r\to0} \frac{r^3\cos^2\theta\sin\theta + r^3\cos\theta\sin^2\theta}{r^2}$

$\lim_{r\to0} r(\cos^2\theta\sin\theta + \cos\theta\sin^2\theta)$

$\lim_{r\to0}$

2

c) $\lim_{r\to0}$ $\lim_{(x,y)\to(0,0)}$ $\frac{r\cos\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$

2

$\lim_{r\to0} \frac{r\cos\theta}{r}$

$= \lim_{r\to0} \frac{r}{\sqrt{r}} \cos\theta$

$-1 \leq \cos\theta \leq 1$

$\frac{-r}{\sqrt{r}} \leq \frac{r}{\sqrt{r}}\cos\theta \leq \frac{r}{\sqrt{r}}$

$\lim_{r\to0} \frac{-r}{\sqrt{r}} = 0$

$\lim_{r\to0} \frac{r}{\sqrt{r}} = 0$

$\therefore \lim_{r\to0} \frac{r}{\sqrt{r}} \cos\theta = 0$

~~$\therefore \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = 0$~~

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4. (20 points) Find an equation for the plane containing the line

$$\vec{r}(t) = (3t-1, -t+4, 5)$$

and the vector $\vec{v} = (1, 1-2)$.

point

Point: (1, 1, -2)

r(0) = (-1, 4, 5)

r(1) = (2, 3, 5)

$\vec{v}_1 = \langle -2, 3, 7 \rangle$

$\vec{v}_2 = \langle 1, 2, 7 \rangle$

$\vec{v}_1 \times \vec{v}_2 = \langle 7, 21, -\cancel{8} \rangle$

$\vec{n} = \langle 7, 21, -8 \rangle$ -7

normal

-2

Plane $0 = \langle 7, 21, -8 \rangle \cdot (\langle x, y, z \rangle - (1, 1, -2))$

Scanned with CamScanner

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5. (20 points) Suppose that a particle in the plane moves with motion $\vec{r}(t)$, velocity $\vec{v}(t) = \vec{r}'(t)$ and acceleration $\vec{a}(t) = \vec{r}''(t)$.

   (a) (10 points) If the particle is moving at constant speed prove that its velocity and acceleration vectors are perpendicular for all times.

   (b) (10 points) If in addition to moving with constant speed its angular momentum $\vec{J}(t) = \vec{r}(t) \times \vec{v}(t)$ is constant, show that for all t, $\vec{v}(t) \perp \vec{r}(t)$ and that the particle is moving on a circle.

a) speed: $s(t) = \int_a^t \|r'(u)\| du \stackrel{?}{=} C$ constant

if $\vec{v}(t)$ and $a(t)$ are $\perp$, $v(t) \cdot a(t) = 0$

$\vec{v}(t) = r'(t)$

$\vec{a}(t) = r''(t)$

Since speed $= \|r'(t)\| = C$

$\|r'(t)\|^2$ also constant

$\|r'(t)\|^2 = r'(t) \cdot r'(t) = c$

$\frac{d}{dt} r'(t) \cdot r'(t) = r'(t) r''(t) + r''(t) r'(t) = v(t) \cdot a(t) + a(t) \cdot v(t)$

$= 2v(t) \cdot a(t)$

$= 0$

because the derivative of a constant $= 0$

$\therefore v(t) \cdot a(t) = 0$, $v(t) \perp a(t)$

b) $J(t) \perp$ to $r(t)$ and $v(t)$

$J'(t) = 0$

$\vec{r}'(t) \times v(t) + r(t) \times v'(t) = 0$

-4