Math 31B
Integration and Infinite Series

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

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Student ID number:

Discussion: 2D

Question	Points	Score
1	11	11
2	10	8
3	10	5
4	9	8
Total:	40	32

Problem 1.

Use u-substitutions and integration by parts to calculate the following indefinite integrals. You might find it better to use the letter t for your substitutions to avoid confusion with your integration by part labels.

(a) [5pts.] $\int \cos\sqrt{x}\, dx$. Hint: the substitution is $t = \sqrt{x}$.

(b) [6pts.] $\int \frac{4\ln(\ln x)\ln x}{x}\, dx$.

a)

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}} dx = \frac{1}{2t} dx$

$t^2 = x$

$\int \cos\sqrt{x}\, dx$

sub t

$= \int (\cos t)(2t)\, dt$ $\quad u = 2t \quad v' = \cos t$

$= 2t\sin t - \int 2\sin t\, dt$ $\quad u' = 2 \quad v = \sin t$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos t + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ // ✓

b)

$\ln x = t$

$\frac{1}{x} dx = dt$

$x = e^t$

$\int \frac{4\ln(\ln x)\ln x}{x} dx$

sub t

$= \int \frac{(4\ln t)(t)}{1} dt$ $\quad u = \ln t \quad v' = t$

$u' = \frac{1}{t} \quad v = \frac{1}{2}t^2$

$= \frac{1}{2}t^2\ln t - \int \frac{1}{t}\frac{t^2}{2} dt$

$= \frac{1}{2}(\ln x)^2\ln(\ln x) - \int \frac{1}{2}t\, dt$

$= \frac{1}{2}(\ln x)^2\ln(\ln x) - \frac{1}{4}t^2 + C$ ✓

$4\left(\frac{1}{2}(\ln x)^2\ln(\ln x) - \frac{1}{4}(\ln x)^2\right) + C$ //

8

Problem 2.

Use L'Hôpital's rule and associated tricks to calculate the following limits.

(a) [5pts.] $\lim_{x\to 1}(1+\ln x)^{\frac{1}{x-1}}$.

(b) [5pts.] $\lim_{x\to 1^+}(\ln x)^2 \ln(\ln x)$.

a)

$y = (1+\ln x)^{\frac{1}{x+1}}$

$\ln y = \frac{1}{x+1} \ln(1+\ln x)$

$\lim_{x\to 1} \ln y = \lim_{x\to 1}\left[\frac{1}{x+1} \ln(1+\ln x)\right]$

$= \frac{1}{2} \ln\left(\lim_{x\to 1}(1+\ln x)\right)$ No.

$= \frac{1}{2} \ln(1)$

$= 0$

-2

$\therefore \lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y}$

$= \lim_{x\to 1} e^{0}$

$= 1$ #

b) $\ln x = t$

$\lim_{x\to 1^+} (\ln x)^2 \ln(\ln x)$

$= \lim_{t\to 0^+} t^2 \ln t$

$= \lim_{t\to 0^+} \frac{\ln t}{t^{-2}}$ ≈ $\frac{\infty}{\infty}$ form, ∴ L'Hôpital

$= \lim_{t\to 0^+} \frac{1/t}{-2t^{-3}}$

$= \lim_{t\to 0^+} \frac{-t^3}{2t}$

$= \lim_{t\to 0^+} -\frac{t^2}{2}$

$= 0$ //

$\sqrt{x} = t$

$dt = \frac{1}{2\sqrt{x}} dx$

Problem 3.

For each of the following improper integrals, either calculate its value, or show it diverges.

4 (a) [6pts.] $\int_1^\infty \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$.

1 (b) [4pts.] $\int_1^e \frac{4\ln(\ln x)\ln x}{x}\,dx$ (looking at previous questions may help).

a)

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}} dx$

$\int_1^\infty \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx$ — limit before u-sub

sub t: $= \int_{\sqrt{1}}^{\sqrt{\infty}} e^t\,dt$

$= \lim_{R\to\sqrt{\infty}} \int_1^R e^t\,dt$

bad

$= \lim_{R\to\sqrt{\infty}} [e^R - e^1]$

$\approx e^{\sqrt{\infty}} - e$

= Does not exist

∴ It diverges.

b)

$t = \ln x$

$dt = \frac{1}{x} dx$

$\int_1^e \frac{4\ln(\ln x)\ln x}{x} dx$ — limits before u-sub

$= \int_{-\infty}^{1} \frac{4\ln(t)\,t}{1} dt$ ✗ 0

$= \lim_{R\to-\infty} \int_R^1 4\ln(t)\,t\,dt$

$u = \ln t \quad v' = 4t$

$u' = \frac{1}{t} \quad v = 2t^2$

$= \lim_{R\to-\infty} \left[2t^2\ln t - \int 2t\,dt\right]_R^1$

$= \lim_{R\to\infty} \left[2t^2\ln t - t^2\right]_R^1$

$= \lim_{R\to-\infty} \left[(-1) - (2R^2\ln R - R^2)\right]$

$= -1 - \lim_{R\to-\infty} \left[R^2(2\ln R - 1)\right]$

$= -1 -$ limit does not exist

= does not exist

∴ It diverges ✗

Problem 4.

(a) [4pts.] Calculate $\lim_{n\to\infty} \frac{n!}{n^n}$.
You should use the squeeze theorem, and justify your inequality similarly to how I did such things in class.

(b) [4pts.] Let $(F_n)_{n=1}^{\infty}$ be the Fibonacci sequence,

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots.$$

It is defined by letting $F_0 = 0$, $F_1 = 1$, and, for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$.
Consider the sequence $(R_n)_{n=1}^{\infty}$,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$$

which is defined by $R_n = \frac{F_{n+1}}{F_n}$.
It is true that (F_n) diverges to ∞, but (R_n) converges to a number L with $1 \leq L \leq 2$.
[You can assume this; there's no need to try and prove it.]
Notice that $\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n}$, so that

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

Prove using the limit laws that $L = \frac{1+\sqrt{5}}{2}$.

(c) [1pts.] Show that $\lim_{n\to\infty} \frac{(2n)!}{n^{2n}} = 0$.
[This is much more difficult; do not waste your time on it if you are already struggling. This is for the A+.]

a)

part (a) answer behind this page

a)

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

$$\left(\frac{1}{n}\right)^n \leq \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot(n-1)\cdot n}{\underbrace{n\cdot n\cdot n\cdot n\cdot\ldots\cdot n\cdot n}_{n\text{-times}}} = \frac{1\cdot 2}{n\cdot n}\cdot\frac{3\cdot 4\cdot\ldots\cdot(n-1)\cdot n}{\underbrace{n\cdot n\cdot\ldots\cdot n\cdot n}_{n-2\text{ times}}}$$

$$= \frac{2}{n^2}\cdot\frac{\left(\frac{n!}{2}\right)}{n^{n-2}} \leq \frac{2}{n^2}\cdot\left(\frac{n^{n-2}}{n^{n-2}}\right)$$

$$= \frac{2}{n^2}\cdot(1)^{n-2}$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^n \leq \lim_{n\to\infty}\frac{n!}{n^n} \leq \lim_{n\to\infty}\frac{2}{n^2}\cdot(1)^{n-2}$$

$$\frac{1}{\lim_{n\to\infty} n^n} \leq \lim_{n\to\infty}\frac{n!}{n^n} \leq \frac{2}{\lim_{n\to\infty} n^2}\cdot 1$$

$$0 \leq \lim_{n\to\infty}\frac{n!}{n^n} \leq 0$$

∴ By squeeze theorem,

$$\lim_{n\to\infty}\frac{n!}{n^n} = 0 \;\#$$

b)

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\left(1 + \frac{F_{n-1}}{F_n}\right)$$

$$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n}$$

$$\lim_{n\to\infty}\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3} \ldots\right\} = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5} \ldots\right\}$$

By observation, $L = 1 + \frac{1}{L}$

$$L^2 = L + 1$$

$$L^2 - L - 1 = 0$$

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

(reject $\frac{1-\sqrt{5}}{2}$ because < 0)

$$= \frac{1+\sqrt{5}}{2}$$ ## shown.

sub $2n = t$

c) $$\lim_{n\to\infty} \frac{(2n)!}{n^{2n}} = \lim_{t\to\infty} \frac{t!}{\left(\frac{t}{2}\right)^t} = \lim_{t\to\infty} \frac{t!}{t^t} \times 2^t$$ (part a shows is $= 0$)

L'hôpital

$$= \lim_{t\to\infty} \frac{2^t}{t^t/t!}$$

$$= \lim_{t\to\infty} \frac{2}{d/dt\,(t^t/t!)} = \frac{\lim_{t\to\infty} 2}{\lim_{t\to\infty} \frac{d}{dt}\left(\frac{t^2}{t!}\right)} \approx \infty$$

(because $\frac{t!}{t^2} \to 0$

$\therefore \frac{t^2}{t!} \to \infty$

$\therefore \frac{d}{dt}\frac{t^2}{t!} \to \infty$ also)

$$= 0$$ ## shown