# Math 31B
## Integration and Infinite Series

## Midterm 1

**Instructions:** You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: HE KAI LIM

Student ID number:

Discussion: 2D

| Question | Points | Score |
|---|---|---|
| 1 | 10 | 10 |
| 2 | 10 | 3 |
| 3 | 10 | 5 |
| 4 | 10 | 9 |
| Total: | 40 | 27 |

**Problem 1.**

4 (a) [4pts.] Calculate $\frac{d}{dx}\left[e^{e^{(x^2+2)}}\right]$

6 (b) [6pts.] Calculate $\int \frac{1}{x(\ln x)^2}\, dx$.

a)

$$\frac{d}{dx}\left(e^{e^{(x^2+2)}}\right) = e^{e^{(x^2+2)}} \times e^{x^2+2} \times 2x$$

$$= 2x\, e^{\left[x^2+2+e^{(x^2+2)}\right]}$$ //.

b) let $\ln x = y$,

$dy = \frac{1}{x}\, dx$

$$\int \frac{1}{x(\ln x)^2}\, dx = \int \frac{1}{y^2}\, dy$$

$$= -y^{-1} + C$$

$$= -\frac{1}{\ln x} + C$$ //.

**Problem 2.**

(a) [3pts.] Let $f(x) = 4 + \frac{3}{1}x + \frac{9}{2}x^2 + \frac{5}{6}x^3 + \frac{11}{24}x^4 + \frac{7}{120}x^5$.
What are the Taylor polynomials $T_3(x)$ and $T_7(x)$ for $f(x)$ centered at 0?

(b) [7pts.] Let $T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots + \frac{(-1)^{n-1}}{n}(x-1)^n$ be the $n$-th Taylor polynomial for $\ln x$ centered at 1.
Find an $n$ such that

$$\left|\ln\left(\frac{1}{2}\right) - T_n\left(\frac{1}{2}\right)\right| < \frac{1}{10^{10}}.$$

a) $a=0$

$$T_3 = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 4 + \frac{\left(\frac{3}{1}\right)}{1}(x) + \frac{2\left(\frac{9}{2}\right)}{2!}(x)^2 + \frac{3\left(\frac{5}{6}\right)(2)}{3!}(x)^3$$

$$= 4 + \frac{3}{1}x + \frac{9}{2}x^2 + \frac{5}{6}x^3$$

$$T_7 = T_3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \frac{f^{(6)}(a)}{6!}(x-a)^6 + \frac{f^{(7)}(a)}{7!}(x-a)^7$$

$$= 4 + \frac{3}{1}x + \frac{9}{2}x^2 + \frac{5}{6}x^3 + \frac{\left(\frac{11}{24}\right)(4)(3)(2)}{4!}x^4 + \frac{\left(\frac{7}{120}\right)(5)(4)(3)(2)}{5!}x^5 + 0 + 0$$

$$= 4 + \frac{3}{1}x + \frac{9}{2}x^2 + \frac{5}{6}x^3 + \frac{11}{24}x^4 + \frac{7}{120}x^5$$

b) For the error bound, it needs condition $|T_{n+1}(x)| \le K$

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \ldots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

For ~~...~~ $\frac{1}{2}$ to 1, $\ln(x)$ is increasing from negative.

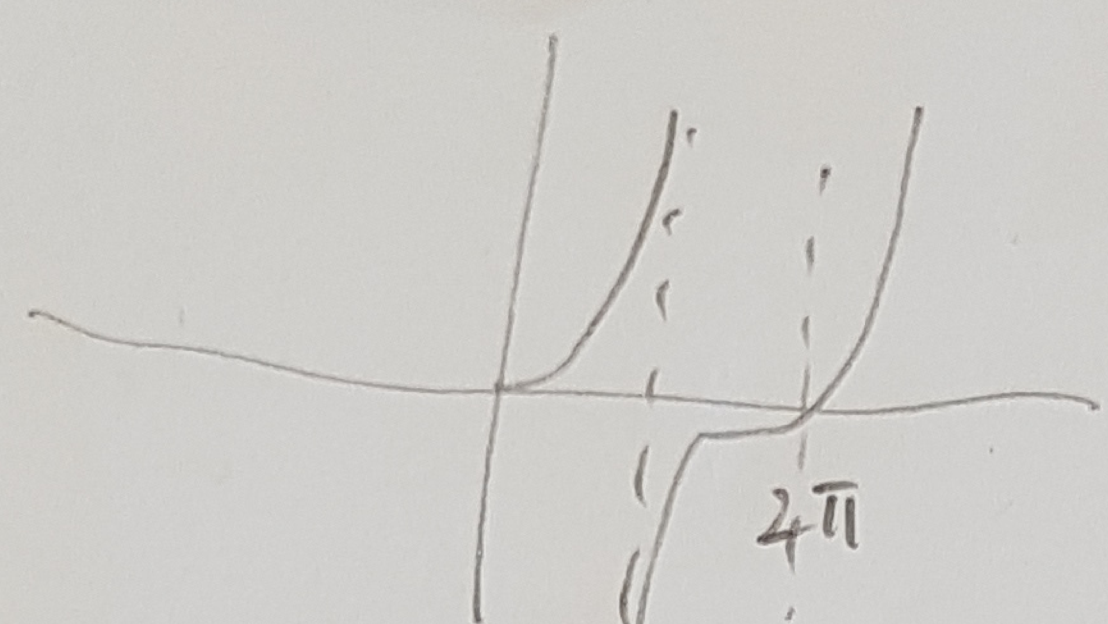
$\therefore |T_{n+1}(x)| \le \ln\frac{1}{2}$

$\therefore K = \ln\frac{1}{2}$

Flip.

$\sin\theta \leq 1$

$\frac{180}{3} = 60$ 5/10



$\frac{1}{\sqrt{3}} \quad 1 \quad \sqrt{3}.$

**Problem 3.**

For (a)-(c), give the value or say, "undefined."

(a) [1pts.] $\tan(\arctan(2)) =$ 2 ✓

(b) [1pts.] $\sin(\arcsin(2)) =$ undefined ✓

(c) [2pts.] $\arctan(\tan(\frac{7\pi}{3})) = \arctan(\tan\frac{\pi}{3}) = \frac{\pi}{3}$ ✓

(d) [6pts.] Suppose $a \neq 0$.

Calculate the following indefinite integral as I did in class (using a $u$-substitution and the knowledge of fundamental integrals which relate to inverse trigonometric functions).

$$\int \frac{1}{a^2 + x^2}\, dx$$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\,dx$$

u-sub?

$$= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x^2}{a}\right)}\,dx$$

$$= \frac{1}{a\cancel{x}} \tan^{-1}\left(\frac{x}{a}\right) + C$$

clox.

+1

9

**Problem 4.** *10pts.*
Calculate the following indefinite integral

$$\int \frac{6x^3+3x^2+9x-8}{(x^2-1)(x^2+4)}\,dx.$$

[The numbers have been chosen so that they work out well; they are all whole numbers.

In the method of partial fractions, I found looking at the $x^3$-coefficient useful.

You'll get points for spotting the correct partial fraction decomposition, and displaying knowledge of the relevant integrals. Notice that you did one of these integrals in 3.d).]

$$\frac{6x^3+3x^2+9x-8}{(x^2-1)(x^2+4)} = \frac{6x^3+3x^2+9x-8}{(x+1)(x-1)(x^2+4)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

$$6x^3+3x^2+9x-8 = A(x-1)(x^2+4) + B(x+1)(x^2+4) + (Cx+D)(x+1)(x-1)$$

let $x=-1$, $\quad -6+3-9-8 = A(-2)(5)$

$-20 = -10A$

$A = 2$

let $x=1$, $\quad 6+3+9-8 = B(2)(5)$

$10 = 10B$

$B = 1$

let $x=0$, $\quad -8 = -4(2) + 4(1) + (-1)D$

$-8 = -4 - D$

$D = 4$

let $x=2$, $\quad 48+12+18-8 = (2)(1)(8) + (1)(3)(8) + 2C(3)(1) + 4(3)(1)$

$70 = 16 + 24 + 6C + 12$

$6C = 18$

$C = 3$

Flip paper

$$\therefore \int \frac{6x^3+3x^2+9x-8}{(x^2-1)(x^2+4)}\,dx = \int \frac{2}{x+1} + \frac{1}{x-1} + \frac{3x+4}{x^2+4}\,dx$$

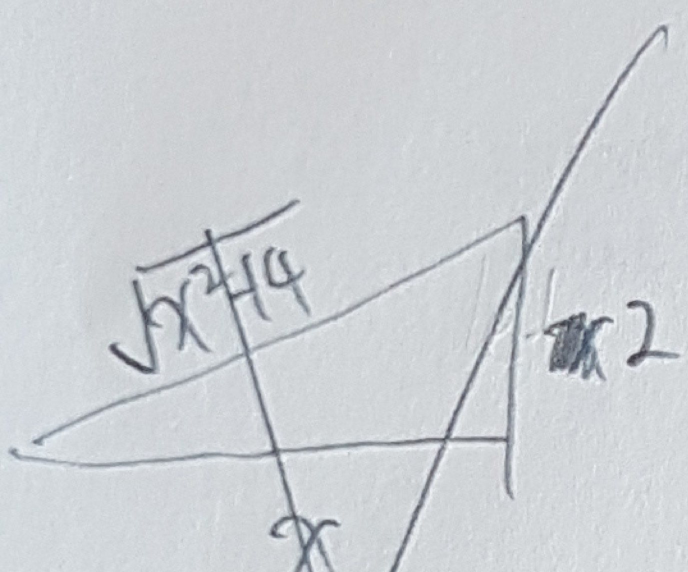
$$= 2\ln|x+1| + \ln|x-1| + \int \frac{3x}{x^2+4} + \frac{4}{x^2+4}\,dx$$

$$= 2\ln|x+1| + \ln|x-1| + \int \frac{1}{(\frac{x}{2})^2+1}\,dx + \int \frac{3x}{x^2+4}\,dx$$

$$= 2\ln|x+1| + \ln|x-1| + \tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{2}\int \frac{2x}{x^2+4}\,dx$$

$$= 2\ln|x+1| + \ln|x-1| + \overset{2}{\tan^{-1}}\left(\frac{x}{2}\right) + \frac{3}{2}\ln|x^2+4| + C$$

$\times, -1$




$\cancel{\ln|x^2+4|}$

$\cancel{\frac{2x}{x^2+4}}$