1. (a) (5 points) Use induction to show that $n! > 2^n$ for any natural number $n \geq 4$.

(*Note that here the base case will be $n = 4$ instead of $n = 1$. Also, $n!$ is notation for $n \times (n-1) \cdots 2 \times 1$.*) 5

10

(b) (5 points) Define a relation $\sim$ on $\mathbb{Z}$ by $a \sim b$ if and only if $a^2 - b^2$ is divisible by 3. Prove that $\sim$ is an equivalence relation. 5

a) We will prove that $n! > 2^n$ for $n \in \mathbb{N}$, $n \geq 4$ with induction. Let $P(n)$ be the statement $n! > 2^n$.

Base case: $P(4)$. We have $4! > 2^4 \iff 24 > 16$ which is true ✓

Inductive Hypothesis: Assume for some $n = K$ that $P(K)$ is true where $K \geq 4$.

Inductive Step: Let us show that if our inductive hypothesis holds, then $P(K+1)$ is also true. Consider $P(K+1) \Rightarrow (K+1)! > 2^{(K+1)}$. Since we know from our Inductive Hypothesis that $P(K) \Rightarrow K! > 2^K \iff 2 \cdot K! > 2^{K+1}$ as $2 > 0$. Since $\frac{K+1}{2} > 1$ clearly for $K \geq 4$, we have that $(K+1)K! > 2^{K+1}$ or $(K+1)! > 2^{K+1}$ as need for $P(K+1)$ to be True. By principle of mathematical induction, we have that $n! > 2^n$ for $n \in \mathbb{N}$, $n \geq 4$.

b) For $\sim$ to be an equivalence Relation, we must have that it is reflexive, symmetric, and transitive.

Symmetry: Consider $a, b \in \mathbb{Z}$ and $a \sim b$. We have that $a^2 - b^2$ is divisible by 3, this means that $(-1) \cdot (a^2 - b^2)$ ~~must~~ $= b^2 - a^2$ must also be divisible by 3. This means $b \sim a$. ✓

Reflexivity: Consider $a \in \mathbb{Z}$. $a^2 - a^2 = 0$ $\forall a \in \mathbb{Z}$. Since 0 is divisible by 3, we have that $\forall a \in \mathbb{Z}$, $a \sim a$. ✓

Transitivity: Consider $a, b, c \in \mathbb{Z}$ and $a \sim b$, $b \sim c$. We have that $a^2 - b^2$ and $b^2 - c^2$ are divisible by 3. It must be then that $(a^2 - b^2) + (b^2 - c^2) = a^2 - c^2$ is also divisible by 3. So $a \sim c$. ✓

Since Reflexivity, symmetry, and transitivity all hold, $\sim$ is an equivalence relation.

2. (a) (2 points) State the definition of the **supremum** of a set of real numbers.

4

(b) (2 points) Let A and B be subsets of the real numbers, $\mathbb{R}$. Prove directly from the definition of supremum that if $A \subset B$, then $\sup A \leq \sup B$.

(c) (3 points) Use the previous part to show that if $A_1 \supset A_2 \supset A_3 \supset \cdots$ is a nested decreasing sequence of sets of **positive** real numbers, then the sequence $\{a_n\}_{n\geq 1}$ given by $a_n = \sup A_n$ is a convergent sequence.

3

(d) (3 points) Find the supremum of the set

$$\left\{(2 - \frac{1}{n+1}) \sin(\frac{n\pi}{8}) \middle| n \in \mathbb{N}\right\},$$

2

and justify how you know that this is the supremum.

a) First let us define an upper bound. An upper bound U_A of a set A of real numbers is defined as some $U_A \in \mathbb{R}$ s.t $\forall a \in A, U_A \geq a$. The supremum is an upper bound of A U_s such that for all other upper bounds U_{A_n} of A, $U_s \leq U_A$. ✓

b) Suppose towards a contradiction that $\sup A > \sup B$. By definition, this means that $\sup B$ is not an upper bound of A. Thus $\exists a \in A$ s.t $a > \sup B$. This however is a contradiction, as $A \subset B$, meaning $\forall a \in A, a \in B$, and $\sup B$ is an upper bound for B so $\forall b \in B, \sup B \geq b$. Thus it must be that $\sup A \leq \sup B$ ✓

c) From part (b), we have that $a_n = \sup A_n \geq a_m = \sup A_m$ if $n \leq m$. This means that $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence. Furthermore, since all A_n are sequences of positive real numbers, we know that $\{a_n\}_{n\geq 1}$ must be bounded below, specifically 0 works as a lower bound. From class we know that every bounded, monotonic sequence must converge, which we have shown $\{a_n\}_{n\geq 1}$ is. ✓

d) Consider the expression $(2-\frac{1}{n+1})\sin(\frac{n\pi}{8}) \leq (2-\frac{1}{n+1})\cdot 1$ (The max sin() can be is 1) $= (2-\frac{1}{n+1}) \leq 2$. Thus we know that 2 is an upper bound for the set. Now let us show that 2 is the supremum. Let ~~be~~ $\varepsilon > 0$, we need to show that $\exists x$ in the set s.t $x > 2-\varepsilon$. In other words, we need to find an $n \in \mathbb{N}$ s.t $2 - \frac{1}{n+1} > 2-\varepsilon$ (we ignore the sin part here as we can assume it is at its max of 1). Since $2>0$, we ~~can~~ need to find $n \in \mathbb{N}$ s.t $-\frac{1}{n+1} < -\varepsilon$, or $\varepsilon < \frac{1}{n+1}$ $(\varepsilon, \frac{1}{n+1} > 0) \iff n < \frac{1}{\varepsilon} - 1$. By Archimedes' principle we know such an n exists. Thus ~~for~~ we can say that 2 is the supremum as no other smaller number can be an upper bound.

for this, need to use the fact that $\sin(\frac{n\pi}{8}) = 1$ infinitely often

3. (a) (2 points) State the definition of convergence.

(b) (4 points) Prove that the sequence $\{a_n\}_{n\geq 1}$ given by $a_n = \frac{5n^2-7n+1}{12n^2-7n+3}$ converges to $\frac{5}{12}$.

(c) (4 points) We say that a sequence $\{a_n\}_{n\geq 1}$ of real numbers is **eventually bounded** if there exists an index $N \in \mathbb{N}$ so that $\{a_n | n \geq N\}$ is a bounded set. Prove that a sequence is eventually bounded if and only if it is bounded.

a) For some sequence $\{a_n\}_{n\geq 1}$ to converge, it must be that $\exists a \in \mathbb{R}$ such that for every $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $n \geq n_\varepsilon \Rightarrow |a - a_n| < \varepsilon$. ✓

b) Let $\varepsilon > 0$. We need to find $n_\varepsilon \in \mathbb{N}$ such that for $n \geq N_\varepsilon$

$$\left|\frac{5n^2-7n+1}{12n^2-7n+3} - \frac{5}{12}\right| < \varepsilon \Leftrightarrow \left|\frac{60n^2-84n+12-60n^2+35n-15}{144n^2-84n+36}\right| < \varepsilon \Leftrightarrow \left|\frac{-49n-3}{144n^2-84n+36}\right| < \varepsilon$$

Since $\left|\frac{-49n-3}{144n^2-84n+36}\right| \leq \frac{49n}{144n^2-84n} = \frac{49}{144n-84}$, if we find n s.t $\frac{49}{144n-84} < \varepsilon$ we have what we need. This means we need to find $n \in \mathbb{N}$ s.t $\frac{49}{\varepsilon} + 84 < 144n$. By archimedes' principle, we know such an n exists, so let n_ε be such an n. Since we chose ε to be arbitrary, we have that $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t $\left|\frac{5n^2-7n+1}{12n^2-7n+3} - \frac{5}{12}\right| < \varepsilon$ or converges to $\frac{5}{12}$ ✓

c) There are 2 cases.

Case 1: $\Rightarrow$: Consider the set $\{a_n | n < N\}$ clearly this set is finite so it must have a max and min value, Thus $\{a_n\}$ is bounded above by $\max(\max(\{a_n | n < N\}),$ upper bound for $\{a_n | n \geq N\})$ and similiarly bounded below. ✓

Case 2: $\Leftarrow$: Clearly let $N = 1$, since $\{a_n\}_{n\geq 1}$ is bounded, $\{a_n | n \geq 1\}$ is also bounded, so $\{a_n\}_{n\geq 1}$ is eventually bounded. ✓

Since both imply the other, the statements are logically equivalent.