115B

The Final

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$(A - 0I)^3 = 0$

This final totals 80 points and you get 180 minutes to do it. Answer as many questions as you can. Good luck!

1. (10 points) Let $T : V \to V$ be a linear map on a vector space V of dimension n over field F. Suppose that V is T-cyclic subspace of itself, show that $f(t) = (-1)^n p(t)$ where $f(t)$ is the characteristic polynomial of T and $p(t)$ is the minimal polynomial of T.

2. (10 points) Let $A \in M_4(F)$ be a 4×4 matrix over a field F. If $A^3 = 0$, up to similarity, write down all possible Jordan canonical forms of A. Explain your answer.

3. (10 points) Find the Jordan canonical form of the matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ where each $a_{ij} = 1$.

4. (10 points) Let V be a dimension 3 inner product space over $\mathbb{R}$. Let $n \geq 1$. Show that the composition of n rotations is a rotation. What is the composition of n reflections in V ? Explain your answer. (You may use the theorem describing orthogonal operators in an m dimensional real inner product space)

5. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

   (a) (3 points) Use the Cayley Hamilton theorem to express A^2 as a linear combination of A and I. Hence express A^{n+2} as a linear combination of A^{n+1} and A^n.

   (b) (3 points) Let $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for ≥ 1. Use part (1) and induction to show $A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$ for all $n \geq 1$

   (c) (4 points) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Show that $F_{n-1} + F_{n+1} = \alpha^n + \beta^n$ and $F_{n-1}F_{n+1} - (F_n)^2 = \alpha^n \beta^n$ for all $n \geq 1$. Actually find $\alpha^n \beta^n$. Explain your answer.

6. (10 points) Let $V = \mathbb{P}_2(\mathbb{R})$, the real vector space of polynomials with real coefficients of degree at most 2.

   (a) (2 points) For $0 \leq i \leq 2$, define $f_i \in V^*$ by $f_i(p(x)) = p(i)$. Show that $\{f_0, f_1, f_2\}$ is a basis of V

   (b) (2 points) Show that there exist unique polynomials $p_0(x), p_1(x), p_2(x)$ such that $p_i(j) = \delta_{ij}$ for $0 \leq i, j \leq 2$.

(c) (3 points) Actually find $p_0(x), p_1(x)$ and $p_2(x)$.

(d) (3 points) Find $q(x) \in V$ such that $q(0) = 0, q(1) = 1$ and $q(2) = 2$. Is such a $q(x)$ unique ? Explain.

7. (10 points) Let T be a normal operator on a complex inner product space V. Show that there exists a normal operator U on V such that $U \circ U = T$. If V is a real inner product space instead, does the same statement still hold? Why or why not ?

8. (10 points) Let T be a linear operator on a finite dimensional real inner product space V. Given that the characteristic polynomial of T splits, show that T is self adjoint. normal

$[T]$ $[T]_\beta$