Mathematics H110B Final Examination

Each problem counts 15 points. Earn at least 100 points including a minimum of 15 points from each of the three parts.

Work on problems done after the three hour exam period is closed book and must be done without any help from notes, books, people, or any other information source. You must write and sign on work turned in after the three hour period: I did all this work by myself without aid from any source.

All rings are commutative.

Part I.

10 → 1. Sketch a proof that a PID is a UFD. (You may assume noetherian rings satisfy any of three equivalent conditions but not a theorem about factorization in noetherian domains.)

15 → 2. Let $R = \mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\}$ a subring of the complex numbers $\mathbb{C}$ with $d \geq 3$ a positive square-free integer. Show that 2 is irreducible but not prime in R. In particular, show that R is not a UFD. ★

15 → 3. Let F be a finite field with q elements. Show the characteristic of F is p for some prime p, F has $q = p^n$ elements for some positive integer n, and every element $\alpha \in F$ satisfies $\alpha^q = \alpha$. ★

4. Let R be a UFD with quotient field F and f and g two primitive polynomials in $R[t]$. Prove that f and g are not relatively prime in $R[t]$ if and only if f and g are not relatively prime in $F[t]$.

15 → 5. Let R be a ring such that for every element $x \in R$ there exists a positive integer $n = n(x) > 1$ such that $x^n = x$. Show that every prime ideal in R is maximal. ★

15 → 6. Let R be a ring and $S \subset R$ a multiplicative set such that $0 \notin S$. We say that S is *saturated* if $ab \in S$ implies that $a, b \in S$. Prove that

(a) S is a saturated multiplicative set if and only if $R \setminus S$ is a union of prime ideals.

(b) The set of zero divisors in a commutative ring is a union of prime ideals.

7. Let R be a domain. If $\mathfrak{m}$ is a maximal ideal in R, let $R_\mathfrak{m} \subset qf(R)$ be the localization of R at $R \setminus \mathfrak{m}$. Prove that

$$R = \bigcap_{\mathfrak{m} \text{ a maximal ideal in } R} R_\mathfrak{m}.$$

★ 8. Let R be a noetherian ring and $\mathfrak{A}$ an irreducible ideal in R, i.e., if $\mathfrak{A}$ is the intersection of two ideals it equals one of them. Prove that $\mathfrak{A}$ is a primary ideal, i.e., if $ab \in \mathfrak{A}$ then either $a \in \mathfrak{A}$ or $\mathfrak{b} \in \sqrt{\mathfrak{A}}$, where $\sqrt{\mathfrak{A}} := \{\mathfrak{b} \in R \mid \mathfrak{b}^n \in \mathfrak{A} \text{ some positive integer } n\}$.

Part II.

1. Let P be an R-module. Then P is called *R-projective* if given any R-epimorphism $f : B \to C$ and R-homomorphism $g : P \to C$, there exists an R-homomorphism $h : P \to B$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

commutes.

Let P be an R-module. Prove the following:

(a) If P is a R-free, then P is R-projective.

(b) P is R-projective if and only if there exists an R-module Q such that $P \coprod Q$ is R-free.

15 2. Let M be an R-module. Show both of the following:

(a) The following are equivalent:

(i) M satisfies DCC (the **decending chain condition**), i.e., if $M_i \subseteq M$ are submodules and

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

then there exists a positive integer N such that $M_N = M_{N+i}$ for all $i \geq 0$.

(ii) M satisfies the **Minimum Principle**, i.e., if $S \neq \emptyset$ is a collection of submodules of M then S contains a minimal element, that is a module $M_o \in S$ such that if $M_o \supseteq N$ with $N \in S$ then $N = M_o$.

[M is called an **artinian R-module** if it satisfies these equivalent conditions.]

(b) If R is a field, then an R-module M is artinian if and only if M is finite dimensional.

3. Let R be a PID and M an R-module generated by n elements. If N is a submodule of M show that it can be generated by $m \leq n$ elements.

15 4. Let R be a noetherian ring and $M \neq 0$ an R-module. Let $S := \{\mathrm{ann}_R(m) \,|\, 0 \neq m \in M\}$. Show that S contains a prime ideal of R.

Part III.

1. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear transformation. Let V be an $F[t]$-module via $tv = T(v)$. Show that V is a finitely generated torsion $F[t]$-module. (You must justify the existence of any special polynomial that arises.)

2. Let $A \in \mathbb{M}_n(\mathbb{Q})$ and $(q_A)' \in \mathbb{Q}[t]$ be the derivative of the minimal polynomial $q_A \in \mathbb{Q}[t]$ of $A \in \mathbb{M}_n(\mathbb{Q})$. Now view $A \in \mathbb{M}_n(\mathbb{C})$. Show A is diagonalizable in $\mathbb{M}_n(\mathbb{C})$, i.e., is similar to a diagonal matrix when viewed in $\mathbb{M}_n(\mathbb{C})$, if and only if the ideal $(q_A, (q_A)')$ in $\mathbb{Q}[t]$ is $\mathbb{Q}[t]$.

3. Describe up to similarity all 3×3 matrices A over the field F satisfying $A^3 = I$ if

(a) F is the complex numbers $\mathbb{C}$.

(b) F is the real numbers $\mathbb{R}$.

(c) F is the finite field $\mathbb{Z}/3\mathbb{Z}$.

4. Compute the Jordan and Rational Canonical Forms of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$